UNIQUENESS IN THE CHARACTERISTIC CAUCHY PROBLEM OF THE KLEIN-GORDON EQUATION AND TAME RESTRICTIONS OF GENERALIZED FUNCTIONS

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Abstract. We show that every tempered distribution, which is a solution of the (homogenous) Klein-Gordon equation, admits a "tame" restriction to the characteristic (hyper)surface $\{x^0 + x^n = 0\}$ in $(1+n)$-dimensional Minkowski space and is uniquely determined by this restriction. The restriction belongs to the space $\mathcal{S}'_{\partial_-}(\mathbb{R}^n)$ which we have introduced in [16]. Moreover, we show that every element of $\mathcal{S}'_{\partial_-}(\mathbb{R}^n)$ appears as the "tame" restriction of a solution of the (homogeneous) Klein-Gordon equation.

1. Introduction

The characteristic Cauchy problem of the Klein-Gordon equation asks for solutions u of the (homogeneous) Klein-Gordon equation $(\Box + m^2)u = 0$, where $\Box = \partial_0^2 - \sum_{i=1}^n \partial_i^2$, with prescribed initial data on the surface $\Sigma = \{x^0 + x^n = 0\}$. Since Σ is a characteristic of the Klein-Gordon operator, the general theory of (linear) partial differential equations predicts non-uniqueness of the solutions unless growth conditions are imposed [12, 11].

The study of the characteristic Cauchy problem is motivated by light cone quantum field theory. In contrast to classical quantum field theory which takes place in Minkowski space-time with coordinates $x = (x^0, x^1, x^2, x^3)$, where x^0 takes on the role of time, light cone quantum field theory uses a different coordinate system obtained by a linear change of variables, where $x^+ = (1/\sqrt{2})(x^0 + x^3)$ is the new time variable The use of this new set of variables, especially the use of x^+ as time (evolution) parameter, results in the use of a new kind of dynamics – called *front form dynamics* – which was introduced by P.A.M. Dirac in [5]. The use of this different kind of dynamics is just the starting point of light cone quantum field theory [3]. Hence, one naturally arrives at the characteristic Cauchy problem when considering fields in the framework of light cone field theory. The problem that the initial data is given on a characteristic surface was widely seen as a big disadvantage of light cone field theory since its beginning. However, Leutwyler et al. [13] assumed that at least within the space of physical solutions uniqueness holds true without giving a proof. Recently, Heinzl and Werner [10] have shown a uniqueness result in the special situation of 1+1 dimensions considering solutions enclosed in a box with various kinds of boundary conditions.

In [16] we have introduced a novel topological vector space $\mathcal{S}_{\partial_-}(\mathbb{R}^n)$ along with its dual space $\mathcal{S}'_{\partial_-}(\mathbb{R}^n)$ in connection with the fundamental problem of light cone quantum field theory that the real scalar free field admits no canonical restriction to $\{x^0 + x^3 = 0\}$. In this paper we will use the space $\mathcal{S}'_{\partial_-}(\mathbb{R}^n)$ to define for each solution $u \in \mathcal{S}'(\mathbb{R}^{1+n})$ of the Klein-Gordon equation $(\Box + m^2)u = 0$ a non-canonical, "tame" restriction $u_0 \in \mathcal{S}'_{\partial_-}(\mathbb{R}^n)$ to the hypersurface Σ in Minkowski space, and show that u is uniquely determined by

u_0. Moreover, we show that each $u_0 \in \mathcal{S}'_{\partial_-}(\mathbb{R}^n)$ appears as the "tame" restriction of some $u \in \mathcal{S}'(\mathbb{R}^{1+n})$ solving the Klein-Gordon equation.

2. Notation and conventions

Let $\mathbb{M} = \mathbb{M}^{1+n}$ denote $(1+n)$-dimensional Minkowski space, i.e., Euclidean space $\mathbb{R}^{1+n}$ together with the bilinear form $\langle x, y\rangle_{\mathbb{M}} = x^0y^0 - \sum_{i=1}^n x^iy^i$. We distinguish the variable x^0 and write $x = (x^0, \mathbf{x}) \in \mathbb{M}$, where $\mathbf{x} = (x^1, \ldots, x^n)$. For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ we denote by $\mathbf{x}\cdot\mathbf{y} = \sum_{i=1}^n x^iy^i$ their Euclidean scalar product, hence $\langle x, y\rangle_{\mathbb{M}} = x^0y^0 - \mathbf{x}\cdot\mathbf{y}$. Furthermore, we set $x^2 = \langle x, x\rangle_{\mathbb{M}}$, $\mathbf{x}^2 = \mathbf{x}\cdot\mathbf{x}$ and $|\mathbf{x}| = \sqrt{\mathbf{x}^2}$. If f is an integrable (complex-valued) function on Minkowski space $\mathbb{M}$, we denote by $\mathcal{F}_{\mathbb{M}}f = f^{\wedge\mathbb{M}}$ the Fourier transform of f with respect to the Minkowski bilinear form, i.e.,$(\mathcal{F}_{\mathbb{M}}f)(p) = f^{\wedge\mathbb{M}}(p) = \int dx f(x)e^{i\langle x,p\rangle_{\mathbb{M}}}$, whereas $\mathcal{F}f = f^\wedge$ denotes the Fourier transform of f with respect to Euclidean scalar product, i.e., $(\mathcal{F}f)(\mathbf{p}) = f^\wedge(\mathbf{p}) = \int d\mathbf{x} f(\mathbf{x})e^{-i\mathbf{x}\cdot\mathbf{p}}$. Recall, that by the inversion formula $f^{\wedge\wedge} = (2\pi)^n f^\vee$, where $f^\vee(x) = f(-x)$.

In the characteristic Cauchy problem of the Klein-Gordon equation the initial data is given on the characteristic surface $\Sigma = \{x^0 + x^n = 0\} \subset \mathbb{M}$. It is appropriate to go over to light-cone coordinates $\tilde{x} = (\tilde{x}^0, \ldots, \tilde{x}^n)$ according to the linear transformation $\tilde{x} = \kappa(x)$ given by $\tilde{x}^0 = (1/\sqrt{2})(x^0 + x^n)$, $\tilde{x}^i = x^i$ $(i = 1, \ldots, n-1)$, $\tilde{x}^n = (1/\sqrt{2})(x^0 - x^n)$. Usually, in physical literature, the components of $\tilde{x}$ are denoted by $\tilde{x} = (x^+, x_\perp, x^-)$, where $x^+ = \tilde{x}^0$, $x_\perp = (\tilde{x}^1, \ldots, \tilde{x}^{n-1})$ and $x^- = \tilde{x}^n$. We will use mainly this notation from physics. We also distinguish the variable x^+ – the LC-time-variable – and write $\tilde{x} = (x^+, \tilde{\mathbf{x}})$, where $\tilde{\mathbf{x}} = (x_\perp, x^-)$. The transformation κ maps Σ onto $\{x^+ = 0\}$. Furthermore, the Minkowski bilinear form is transformed to the LC-bilinear form $\langle \tilde{x}, \tilde{y}\rangle_{\mathbb{L}} = x^+y^- + x^-y^+ - x_\perp \cdot y_\perp$ by κ, where $x_\perp \cdot y_\perp = \sum_{i=1}^{n-1} \tilde{x}^i\tilde{y}^i$. We set $\tilde{x}^2 = \langle \tilde{x}, \tilde{x}\rangle_{\mathbb{L}} = 2x^+x^- - x_\perp \cdot x_\perp$. We denote $\mathbb{L} = \mathbb{L}^{1+n}$ the bilinear space consisting of $\mathbb{R}^{1+n}$ and $\langle ., .\rangle_{\mathbb{L}}$, and call it $(1+n)$-dimensional *LC-space*. Hence, $\kappa : \mathbb{M} \xrightarrow{\sim} \mathbb{L}$ is an isomorphism of bilinear spaces. If f is an integrable (complex-valued) function on LC-space $\mathbb{L}$, we denote by $\mathcal{F}_{\mathbb{L}}f = f^{\wedge\mathbb{L}}$ the Fourier transform of f with respect to the LC-bilinear form, i.e., $(\mathcal{F}_{\mathbb{L}}f)(\tilde{p}) = f^{\wedge\mathbb{L}}(\tilde{p}) = \int d\tilde{x} f(\tilde{x})e^{i\langle \tilde{x},\tilde{p}\rangle_{\mathbb{L}}}$. Notice that κ commutes with these Fourier transformations, i.e., $(f \circ \kappa)^{\wedge\mathbb{M}} = f^{\wedge\mathbb{L}} \circ \kappa$. Next we need to introduce a further Fourier transformation which affects only the spatial part of $\tilde{x} = (x^+, \tilde{\mathbf{x}})$. Since in the LC-bilinear form $\langle x, p\rangle_{\mathbb{L}} = x^+p^- + x^-p^+ - x_\perp \cdot p_\perp$ the time-variable x^+ is paired with p^-, the variable p^- will be considered as energy-variable in physical literature. Hence the variable $p = (p^+, p_\perp, p^-)$ is split into $p = (\tilde{\mathbf{p}}, p^-)$, where $\tilde{\mathbf{p}} = (p^+, p_\perp)$ denotes the spatial momentum. In contrast, we split x into $x = (x^+, \tilde{\mathbf{x}})$, since we have distinguished x^+ as time-variable. Here, a little bit care is needed. If $f : \mathbb{R}^n \to \mathbb{C}$ is an (absolutely) integrable function, we set

$$(\mathcal{F}_{\mathbb{L}}^{\tilde{\mathbf{x}}\to\tilde{\mathbf{p}}} f)(\tilde{\mathbf{p}}) = f^{\sqcap}(\tilde{\mathbf{p}}) = \int d^n\tilde{\mathbf{x}} f(\tilde{\mathbf{x}})e^{i(x^-p^+ - x_\perp\cdot p_\perp)}. \tag{2.1}$$

Let $\mathcal{S}(\mathbb{R}^n)$ denote the Schwartz space consisting of rapidly decreasing, smooth, complex-valued functions on $\mathbb{R}^n$, i.e., complex-valued C^∞-functions f on $\mathbb{R}^n$ such that $\sup(1 + |\mathbf{x}|)^N|\partial^\alpha f(\mathbf{x})| < \infty$ for all $N \in \mathbb{N}$ and multi-indices α. $\mathcal{S}(\mathbb{R}^n)$ is topologized by the family of seminorms $\sup(1 + |\mathbf{x}|)^N|\partial^\alpha.|$ $(N \in \mathbb{N}, \alpha$ multi-index$)$. The dual space $\mathcal{S}'(\mathbb{R}^n)$ is called the space of generalized functions (or tempered distributions). Usually, $\mathcal{S}'(\mathbb{R}^n)$ carries the weak*-topology. There is also a canonical embedding $\mathcal{S}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$ and, in the

sequel, we often identify $\mathcal{S}(\mathbb{R}^n)$ with its image, i.e., we assume $\mathcal{S}(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$. As is well known (cf. e.g. [14]), the Fourier transformation $\mathcal{F}$ is a linear homeomorphism from $\mathcal{S}(\mathbb{R}^n)$ onto $\mathcal{S}(\mathbb{R}^n)$ which extends to a linear, sequentially continuous mapping from $\mathcal{S}'(\mathbb{R}^n)$ onto $\mathcal{S}'(\mathbb{R}^n)$. Obviously, the same holds for $\mathcal{F}_{\mathbb{M}}$, $\mathcal{F}_{\mathbb{L}}$ and $\mathcal{F}_{\mathbb{L}}^{\tilde{\mathbf{x}}\to\tilde{\mathbf{p}}}$. We denote by $\mathcal{D}(U)$ ($U \subset \mathbb{R}^n$) the topological vector space of all complex-valued smooth, i.e., C^∞ functions on U with compact support, and by $\mathcal{D}'(U)$ its dual space – the space of distributions [11, 14].

Furthermore, we set $\Gamma_m^\pm = \{p \in \mathbb{M}^{1+n} : p^2 - m^2 = 0,\ \pm p^0 > 0\}$, $\widetilde{\Gamma}_m^\pm = \{\tilde{p} \in \mathbb{L}^{1+n} : \tilde{p}^2 - m^2 = 0,\ \pm p^- > 0\}$ and $\Omega_\pm : \mathbb{R}^n \to \Gamma_m^\pm$, $\mathbf{p} \mapsto (\pm\omega(\mathbf{p}), \mathbf{p})$, $\omega(\mathbf{p}) = \sqrt{\mathbf{p}^2 + m^2}$ and $\widetilde{\Omega}_\pm : \mathbb{R}_{\gtrless 0} \times \mathbb{R}^{n-1} \to \widetilde{\Gamma}_m^\pm$, $\tilde{\mathbf{p}} \mapsto (\tilde{\mathbf{p}}, \tilde{\omega}(\tilde{\mathbf{p}}))$, $\tilde{\omega}(\tilde{\mathbf{p}}) = (\tilde{\mathbf{p}}, \tilde{\omega}(\tilde{\mathbf{p}}))$, $\tilde{\omega}(\tilde{\mathbf{p}}) = (1/2p^+)(p_\perp^2 + m^2)$, and $\widetilde{\Omega}(\tilde{\mathbf{p}}) = (\tilde{\mathbf{p}}, \tilde{\omega}(\tilde{\mathbf{p}}))$ ($\tilde{\mathbf{p}} \in \mathbb{R}^n \setminus \{p^+ = 0\}$).

3. Review of the non-characteristic Cauchy problem

It is well known (see, e.g., [2]) that the following (non-characteristic) Cauchy problem of the Klein-Gordon equation, stated in $\mathcal{S}'(\mathbb{R}^{1+n})$, is well-posed. Let $\Box = \partial_0^2 - \Delta_{\mathbf{x}}$ denote the d'Alembert operator. Then a solution $u \in \mathcal{S}'(\mathbb{R}^{1+n})$ of

$$(\Box + m^2)u = 0, \qquad u|_{x^0=0} = u_0, \qquad \partial_0 u|_{x^0=0} = u_1. \tag{3.1}$$

exists and is unique for any $u_0, u_1 \in \mathcal{S}'(\mathbb{R}^n)$. Since in the following sections we essentially make use of a special representation of the solutions of (3.1), we will give a short review of the non-characteristic Cauchy problem (3.1). First of all, we have to note that any solution $u \in \mathcal{S}'(\mathbb{R}^{1+n})$ of the Klein-Gordon equation is C^∞-dependent on $x^0 \in \mathbb{R}$ as a parameter [2], and hence admits a restriction $u|_{x^0=0}$ to $\{x^0 = 0\}$. This follows easily from the fact that the Klein-Gordon operator is hypoelliptic with respect to x^0 [7], [8], [6]. Thus one obtains a uniquely determined family $(u_{x^0})_{x^0\in\mathbb{R}}$ in $\mathcal{S}'(\mathbb{R}^n)$ such that

$$(u(x), f(x^0)g(\mathbf{x})) = \int (u_{x^0}, g) f(x^0) dx^0 \tag{3.2}$$

for all $f(x^0) \in \mathcal{S}(\mathbb{R})$, $g(\mathbf{x}) \in \mathcal{S}(\mathbb{R}^n)$. The restriction $u|_{x^0=0}$ is then, per definition, $u_{x^0=0}$. Usually, the uniqueness of a solution of (3.1) is proven by showing that the (parameter) derivatives of any order of u_{x^0} with respect to x^0 vanish at $x^0 = 0$, and that the family $(u_{x^0})_{x^0\in\mathbb{R}}$ depend analytically on x^0 (the last assertion follows by a theorem of Paley and Wiener [14]). These arguments are not directly applicable to the characteristic Cauchy problem. Hence we will reprove uniqueness and existence of (3.1) in a manner which is more in the spirit of the proof of uniqueness and existence of the characteristic Cauchy problem. Moreover, we need the following results as a preparation for studying the connection between the characteristic and the non-characteristic Cauchy problem in Subsection 5.2.

Definition 3.1. For each $a(\mathbf{p}) \in \mathcal{S}'(\mathbb{R}^n)$ we define the generalized functions $a(\mathbf{p})\delta_\pm(p^2 - m^2) \in \mathcal{S}'(\mathbb{R}^{1+n})$ by

$$(a(\mathbf{p})\delta_\pm(p^2 - m^2), f(p)) = \left(a(\mathbf{p}), \frac{f(\Omega_\pm(\mathbf{p}))}{2\omega(\mathbf{p})}\right), \qquad (f(p) \in \mathcal{S}(\mathbb{R}^{1+n})),$$

where $\Omega_\pm(\mathbf{p}) = (\pm\omega(\mathbf{p}), \mathbf{p})$ and $\omega(\mathbf{p}) = \sqrt{\mathbf{p}^2 + m^2}$ ($\mathbf{p} \in \mathbb{R}^n$).

Remark 3.2. a) Let $\Gamma_m = \{p \in \mathbb{R}^{1+n} : p^2 - m^2 = 0\}$ be the mass-hyperboloid. Then the projection map $\Gamma_m \to \mathbb{R}^n$, $p = (p^0, \mathbf{p}) \mapsto \mathbf{p}$ is a double covering of $\mathbb{R}^n$. The restriction of the projection map to each of the connected components $\Gamma_m^\pm = \Gamma_m \cap \{\pm p^0 > 0\}$ of Γ_m is a homeomorphism $\Gamma_m^\pm \xrightarrow{\sim} \mathbb{R}^n$ whose inverse mapping is $\mathbb{R}^n \to \Gamma_m^\pm$, $\mathbf{p} \mapsto \Omega_\pm(\mathbf{p})$.

b) Since $f(p) \mapsto f(\Omega_\pm(\mathbf{p}))/2\omega(\mathbf{p})$ are continuous, linear maps from $\mathcal{S}(\mathbb{R}^{1+n})$ to $\mathcal{S}(\mathbb{R}^n)$, $a(\mathbf{p})\delta_\pm(p^2 - m^2)$ are well-defined tempered distributions.

c) We denote by dS the canonical surface measure on Γ_m. If $a(\mathbf{p}) \in \mathcal{S}(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$ then

$$(a(\mathbf{p})\delta_\pm(p^2 - m^2), f(p)) = \int_{\Gamma_m^\pm} \frac{(a \circ \Omega_\pm^{-1})(p) f(p)}{|\nabla Q(p)|} dS(p), \qquad (f(p) \in \mathcal{S}(\mathbb{R}^{1+n})),$$

where $Q(p) = p^2 - m^2$.

d) The support of $a(\mathbf{p})\delta_\pm(p^2 - m^2)$ is contained in $\Gamma_m^\pm$.

The following lemma is an immediate consequence of the definition.

Lemma 3.3. *The mappings $\mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^{1+n})$, $a(\mathbf{p}) \mapsto a(\mathbf{p})\delta_\pm(p^2 - m^2)$ are $\mathbb{C}$-linear and sequentially continuous.*

Lemma 3.4. *Let $a_0(\mathbf{p}), a_1(\mathbf{p}), a(\mathbf{p}) \in \mathcal{S}'(\mathbb{R}^n)$.*

(i) $a(\mathbf{p})\delta_\pm(p^2 - m^2) = 0$ if and only if $a(\mathbf{p}) = 0$.

(ii) $a_0(\mathbf{p})\delta_+(p^2 - m^2) + a_1(\mathbf{p})\delta_-(p^2 - m^2) = 0$ if and only if $a_0(\mathbf{p}) = a_1(\mathbf{p}) = 0$.

Proof. (i) This follows from the fact, that the maps

$$\mathcal{S}(\mathbb{R}^{1+n}) \to \mathcal{S}(\mathbb{R}^n),\ f(p) \mapsto f(\Omega_\pm(\mathbf{p}))/2\omega(\mathbf{p})$$

are onto. For, given $g(\mathbf{p}) \in \mathcal{S}(\mathbb{R}^n)$ then $f(p) = 2\omega(\mathbf{p})e^{-(p^0 \mp \omega(\mathbf{p}))^2} g(\mathbf{p}) \in \mathcal{S}(\mathbb{R}^{1+n})$ does the job.

(ii) From $a_0(\mathbf{p})\delta_+(p^2 - m^2) + a_1(\mathbf{p})\delta_-(p^2 - m^2) = 0$ it follows $a_0(\mathbf{p})\delta_+(p^2 - m^2) = -a_1(\mathbf{p})\delta_-(p^2 - m^2)$. Now, since the supports are disjoint (cf. Remark 3.2 (d)), the assertion follows from (i). $\square$

Proposition 3.5. *The general solution of the division problem $(p^2 - m^2)u = 0$, $u \in \mathcal{S}'(\mathbb{R}^{1+n})$ is given by*

$$u(p) = a_0(\mathbf{p})\delta_+(p^2 - m^2) + a_1(\mathbf{p})\delta_-(p^2 - m^2),$$

where $a_0(\mathbf{p}), a_1(\mathbf{p}) \in \mathcal{S}'(\mathbb{R}^n)$. Moreover, $a_0(\mathbf{p})$ and $a_1(\mathbf{p})$ are uniquely determined by $u(p)$.

Proof. Uniqueness follows from Lemma 3.4. To prove existence (see also e.g. [2], p. 60) let $u \in \mathcal{S}'(\mathbb{R}^{1+n})$ be a solution of the division problem. Since $\mathrm{supp}(u) \subset \Gamma_m$ and Γ_m has the two connected components $\Gamma_m^\pm = \{p \in \Gamma_m : \pm p^0 > 0\}$ we can uniquely split $u = u_+ + u_-$ with $u_\pm = u|_{\pm p^0 > 0}$. Hence in the following we may assume w.l.o.g. that $u = u_+$, i.e., $\mathrm{supp}(u) \subset \Gamma_m^+$. Consider the smooth coordinate transformation

$$\lambda : \mathbb{R}_{>0} \times \mathbb{R}^n, (p^0, \mathbf{p}) \mapsto (t, \mathbf{p}) = (p^2 - m^2, \mathbf{p})$$

which maps Γ_m^+ onto $\{0\} \times \mathbb{R}^n$. Since u has support in Γ_m^+ $\lambda_* u = u \circ \lambda^{-1}$ has support in $\{0\} \times \mathbb{R}^n$. Hence we can write $\lambda_* u = \delta(t) \otimes a_0(\mathbf{p})$ with some $a_0(\mathbf{p}) \in \mathcal{S}'(\mathbb{R}^n)$, and thus

$u = \lambda^*(\delta(t) \otimes a_0(\mathbf{p}))$. By the general formula for smooth coordinate changes (see, e.g., [14, 11]) we obtain

$$\begin{aligned}
(\lambda^*(\delta(t) \otimes a_0(\mathbf{p})), f) &= (\delta(t) \otimes a_0(\mathbf{p}), \frac{f(\sqrt{t + \mathbf{p}^2 + m^2}, \mathbf{p})}{2\sqrt{t + \mathbf{p}^2 + m^2}}) \\
&= (a_0(\mathbf{p}), \frac{f(\omega(\mathbf{p}), \mathbf{p})}{2\omega(\mathbf{p})}) \\
&= (a_0(\mathbf{p})\delta_+(p^2 - m^2), f)
\end{aligned}$$

for all $f \in \mathcal{S}(\mathbb{R}^{1+n})$. $\square$

Proposition 3.6. *Assume $a_0(\mathbf{p}), a_1(\mathbf{p}) \in \mathcal{S}'(\mathbb{R}^n)$ and let $u(x) = \mathcal{F}_{\mathbb{M}}^{-1}(a_0(\mathbf{p})\delta_+(p^2 - m^2) + a_1(\mathbf{p})\delta_-(p^2 - m^2)) \in \mathcal{S}'(\mathbb{R}^{1+n})$. Then $u(x) = u(x^0, \mathbf{x})$ is C^∞-dependent on x^0 as a parameter, and for all $x^0 \in \mathbb{R}$*

$$\begin{aligned}
u_{x^0} &= \frac{1}{4\pi}\mathcal{F}^{-1}\left(\frac{a_0(\mathbf{p})e^{-i\omega(\mathbf{p})x^0} + a_1(\mathbf{p})e^{i\omega(\mathbf{p})x^0}}{\omega(\mathbf{p})}\right), \\
(\partial_0 u)_{x^0} &= \frac{1}{4\pi i}\mathcal{F}^{-1}\left(a_0(\mathbf{p})e^{-i\omega(\mathbf{p})x^0} - a_1(\mathbf{p})e^{i\omega(\mathbf{p})x^0}\right).
\end{aligned}$$

Proof. We define the family $(u_{x^0})_{x^0 \in \mathbb{R}}$ by $u_{x^0} = \frac{1}{4\pi}\mathcal{F}^{-1}\left(\frac{a_0(\mathbf{p})e^{-i\omega(\mathbf{p})x^0} + a_1(\mathbf{p})e^{i\omega(\mathbf{p})x^0}}{\omega(\mathbf{p})}\right) \in \mathcal{S}'(\mathbb{R}^n)$. Let $f(x^0) \in \mathcal{S}(\mathbb{R})$ and $g(\mathbf{x}) \in \mathcal{S}(\mathbb{R}^n)$. We have to show that $u(\mathbf{x})$ fulfills the equation (3.2).

Case 1: Firstly, we assume $a_0(\mathbf{p}), a_1(\mathbf{p}) \in \mathcal{S}(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$. Then, by an easy computation, we obtain

$$\begin{aligned}
(u, f \otimes g) &= \big(a_0(\mathbf{p})\delta_+(p^2 - m^2) + a_1(\mathbf{p})\delta_-(p^2 - m^2), \mathcal{F}_{\mathbb{M}}^{-1}(f \otimes g)\big) \\
&= \frac{1}{(2\pi)^{n+1}} \int dx^0 f(x^0) \int d^n\mathbf{x} g(\mathbf{x}) \int \frac{d^n\mathbf{p}}{2\omega(\mathbf{p})} \left(a_0(\mathbf{p})e^{-i\omega(\mathbf{p})x^0} + a_1(\mathbf{p})e^{i\omega(\mathbf{p})x^0}\right) e^{i\mathbf{p}\cdot\mathbf{x}} \\
&= \frac{1}{2\pi} \int dx^0 f(x^0) \int \frac{d^n\mathbf{p}}{2\omega(\mathbf{p})} \left(a_0(\mathbf{p})e^{-i\omega(\mathbf{p})x^0} + a_1(\mathbf{p})e^{i\omega(\mathbf{p})x^0}\right) (\mathcal{F}^{-1}g)(\mathbf{p}) \\
&= \int dx^0 f(x^0)(u_{x^0}, g).
\end{aligned}$$

Case 2: Now, assume $a_0(\mathbf{p}), a_1(\mathbf{p}) \in \mathcal{S}'(\mathbb{R}^n)$. Choose sequences $(a_0^{(m)}(\mathbf{p}))$ and $(a_1^{(m)}(\mathbf{p}))$ in $\mathcal{S}(\mathbb{R}^n)$ such that $a_0^{(m)}(\mathbf{p}) \to a_0(\mathbf{p})$, $a_1^{(m)}(\mathbf{p}) \to a_1(\mathbf{p})$ in $\mathcal{S}'(\mathbb{R}^n)$ and define $u^{(m)} = \mathcal{F}_{\mathbb{M}}^{-1}(a_0^m(\mathbf{p})\delta_+(p^2 - m^2) + a_1^m(\mathbf{p})\delta_-(p^2 - m^2)) \in \mathcal{S}'(\mathbb{R}^{1+n})$. Then $u^{(m)}$ converges to u in $\mathcal{S}'(\mathbb{R}^{1+n})$ and, by construction, $u_{x^0}^{(m)}$ converges to u_{x^0} in $\mathcal{S}'(\mathbb{R}^n)$ $(m \to \infty)$. By the first case, we have

$$(u, f \otimes g) = \lim_{m\to\infty} \int (u_{x^0}^{(m)}, g) f(x^0) dx^0. \tag{3.3}$$

Since $|(u_{x^0}^{(m)}, g)| \leq (4\pi)^{-1} \int \frac{d\mathbf{p}}{\omega(\mathbf{p})}(|(a_0^{(m)}(\mathbf{p})| + |a_1^{(m)}(\mathbf{p})|)|\mathcal{F}^{-1}g(\mathbf{p})| = C < \infty$ where the constant C doesn't depend on x^0, the right-hand side of (3.3) equals $\int (u_{x^0}, g) f(x^0) dx^0$ by dominant convergence. $\square$

Corollary 3.7. *Any solution $u \in \mathcal{S}'(\mathbb{R}^{1+n})$ of the Klein-Gordon equation $(\Box + m^2)u = 0$ is C^∞-dependent on $x^0 \in \mathbb{R}$ as a parameter.*

Corollary 3.8. *The (non-characteristic) Cauchy problem* (3.1) *has a unique solution for any $u_0, u_1 \in \mathcal{S}'(\mathbb{R}^n)$.*

Proof. Assume $u_0(\mathbf{x}), u_1(\mathbf{x}) \in \mathcal{S}'(\mathbb{R}^n)$. Define $a_0(\mathbf{p}), a_1(\mathbf{p}) \in \mathcal{S}'(\mathbb{R}^n)$ by

$$a_0(\mathbf{p}) = 2\pi\left(\omega(\mathbf{p})\hat{u}_0(\mathbf{p}) + i\hat{u}_1(\mathbf{p})\right) \quad \text{and} \quad a_1(\mathbf{p}) = 2\pi\left(\omega(\mathbf{p})\hat{u}_0(\mathbf{p}) - i\hat{u}_1(\mathbf{p})\right).$$

Then $u = \mathcal{F}_{\mathbb{M}}^{-1}(a_0(\mathbf{p})\delta_+(p^2-m^2)+a_1(\mathbf{p})\delta_-(p^2-m^2))$ is a solution of (3.1) by Proposition 3.6. Uniqueness follows immediately from Propositions 3.5 and 3.6. $\Box$

4. Squeezed generalized functions and tame restrictions

4.1. **Definitions and elementary properties.** In this subsection we introduce squeezed generalized functions and show some general properties of this class of functions which will be important in the sequel. Since we have already introduced squeezed generalized functions in [16], we only give a summary of results and omit most of the proofs.

Definition 4.1. (a) Let $\mathcal{S}_{p^+}(\mathbb{R}^n)$ be the set of all $f \in C^\infty(\mathbb{R}^n, \mathbb{C})$ such that

$$||f||_{k,\beta,\alpha} = \sup_{(p^+,p_\perp)\in\mathbb{R}^n\setminus\{p=0\}} |(p^+)^k p_\perp^\beta \partial^\alpha f(p^+, p_\perp)| < \infty \tag{4.1}$$

for all $k \in \mathbb{Z}$ and all multi-indices α, β. We endow $\mathcal{S}_{p^+}(\mathbb{R}^n)$ with the locally convex topology defined by the seminorms $||\_\_||_{k,\beta,\alpha}$ and call it the *squeezed Schwartz space* (of squeezed rapidly decreasing functions). The dual space $\mathcal{S}'_{p^+}(\mathbb{R}^n)$ is called the space of *squeezed generalized functions* (or *squeezed tempered distributions*).

(b) We call $\mathcal{S}_{p^+\gtrless 0}(\mathbb{R}^n) = \{f \in \mathcal{S}_{p^+}(\mathbb{R}^n) : f|_{p^+\lessgtr 0} \equiv 0\}$ the *positive/negative squeezed Schwartz space* (of positive/negative squeezed rapidly decreasing functions). The dual space $\mathcal{S}'_{p^+\gtrless 0}(\mathbb{R}^n)$ is called the space of *positive/negative squeezed generalized functions*.

(c) On $\mathcal{S}'_{p^+}(\mathbb{R}^n)$ and $\mathcal{S}'_{p^+\gtrless 0}(\mathbb{R}^n)$ we consider always the weak*-topology, i.e., the locally convex topology defined by the family of seminorms $u \mapsto |u(f)|$, $f \in \mathcal{S}_{p^+}(\mathbb{R}^n)$, respectively, $f \in \mathcal{S}_{p^+\gtrless 0}(\mathbb{R}^n)$.

In the following we denote by $\mathbb{C}[p^+, p_\perp]_{p^+} = S^{-1}\mathbb{C}[p^+, p_\perp]$ the localization (cf. [1]) of the polynomial ring $\mathbb{C}[p^+, \tilde{p}_\perp]$ by the multiplicative set $S = \{(p^+)^k : k \geq 0\}$. It consists of formal fractions $\frac{R}{(p^+)^k}$ where $R \in \mathbb{C}[p^+, p_\perp]$, $k \in \mathbb{N}$. Alternatively, we can describe the elements of $\mathbb{C}[p^+, p_\perp]_{p^+}$ as Laurent polynomials in p^+, i.e., each element of $\mathbb{C}[p^+, p_\perp]_{p^+}$ is of the form $T(p^+, p_\perp, 1/p^+)$, where $T(p^+, p_\perp, t) \in \mathbb{C}[p^+, p_\perp, t]$ is a "usual" polynomial. We consider the elements of $\mathbb{C}[p^+, \tilde{p}_\perp]_{p^+}$ as $\mathbb{C}$-valued functions on $\mathbb{C}^n \setminus \{p^+ = 0\}$. Recall that in algebraic geometry [9] $\mathbb{C}[p^+, p_\perp]_{p^+}$ is called the ring of regular functions on the (distinguished) open set $D(p^+) = \mathbb{C}^n \setminus \{p^+ = 0\}$.

Remark 4.2. The space $\mathcal{S}_{p^+}(\mathbb{R}^n)$ can also be defined by the following seminorms which are equivalent to the seminorms in (4.1):

$$||f||_{Q,\alpha} = \sup_{\tilde{\mathbf{p}}\in\mathbb{R}^n\setminus\{p^+=0\}} |Q(\tilde{\mathbf{p}})\partial^\alpha f(\tilde{\mathbf{p}})|, \qquad Q \in \mathbb{C}[p^+, p_\perp]_{p^+},\ \alpha \text{ multi-index},$$

or

$$||f||_{N,\alpha} = \sup_{\tilde{\mathbf{p}}\in\mathbb{R}^n\setminus\{p^+=0\}} \left(\frac{1+|\tilde{\mathbf{p}}|}{|p^+|}\right)^N |\partial^\alpha f(\tilde{\mathbf{p}})|, \qquad N\in\mathbb{N},\ \alpha \text{ multi-index.}$$

Remark 4.3. 1) Let $f \in \mathcal{S}_{p^+}(\mathbb{R}^n)$. From $||f||_{-k,0,\alpha} = C < \infty$ it follows $|\partial^\alpha f(\tilde{\mathbf{p}})| \leq C|(p^+)^k|$ ($k \in \mathbb{N}$, α multi-index). Hence, as p^+ goes to 0, any partial derivative of f goes faster to 0 than any power of p^+. Since f is a C^∞-function, $\partial^\alpha f|_{\{p^+=0\}} = 0$. Moreover, any $f \in \mathcal{S}_{p^+}(\mathbb{R}^n)$ has the same rapidly decreasing behavior (as $|\tilde{\mathbf{p}}|$ goes to infinity) as the functions of the Schwartz space $\mathcal{S}(\mathbb{R}^n)$. More precisely, we have $\mathcal{S}_{p^+}(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$.

2) If f is a (complex-valued) C^∞-function on $\mathbb{R}^n \setminus \{p^+ = 0\}$ such that $||f||_{k,\alpha,\beta} < \infty$ for all $k \in \mathbb{Z}$ and all multi-indices α, β then one can easily verify that f has a unique continuous extension to $\{p^+ = 0\}$ – necessarily $f(p^+ = 0) = 0$. This continuous extension of f is a C^∞-function on $\mathbb{R}^n$, and hence belongs to $\mathcal{S}_{p^+}(\mathbb{R}^n)$. In the following we always consider in such a case the extension (by zero) of f (to $\{p^+ = 0\}$), also denoted f, without mentioning it explicitly.

One of the reasons why squeezed generalized functions are so important for us is the fact that $\mathcal{S}_{p^+>0}(\mathbb{R}^n)$ (as well as $\mathcal{S}_{p^+<0}(\mathbb{R}^n)$) is the $\mathbb{C}$-linear homeomorphic image of the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ under a certain map called the *squeezing mapping.*

Definition 4.4. We call the mapping $\nu_{\gtrless 0} : \mathbb{R}_{\gtrless 0}\times\mathbb{R}^{n-1} \to \mathbb{R}^n$ defined by $\nu_{\gtrless 0} = \Omega_\pm^{-1}\circ\kappa^{-1}\circ\widetilde{\Omega}_\pm$ the *positive/negative squeezing mapping.*

Proposition 4.5. *The mapping $f \mapsto j(\nu_{\gtrless 0}^* f) = j(f \circ \nu_{\gtrless 0})$, where $j(.)$ denotes extension by zero, defines a linear homeomorphism from $\mathcal{S}(\mathbb{R}^n)$ onto $\mathcal{S}_{p^+\gtrless 0}(\mathbb{R}^n)$.*

Proof. cf. [16]. □

By abuse of language we neglect j in the notation and write simply $\nu_{\gtrless 0}^*$ instead of $j(\nu_{\gtrless 0}^*)$, hence $\nu_{\gtrless 0}^* : \mathcal{S}(\mathbb{R}^n) \xrightarrow{\sim} \mathcal{S}_{p^+\gtrless 0}(\mathbb{R}^n)$ is an isomorphism of (complex) topological vector spaces.

Theorem 4.6. *(i) $\mathcal{S}_{p^+}(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$, and the topology of $\mathcal{S}_{p^+}(\mathbb{R}^n)$ coincides with the subspace topology induced by $\mathcal{S}(\mathbb{R}^n)$. Moreover, $\mathcal{S}_{p^+}(\mathbb{R}^n)$ is a closed subspace of $\mathcal{S}(\mathbb{R}^n)$.*

(ii) $\mathcal{S}_{p^+}(\mathbb{R}^n) = \mathcal{S}_{p^+>0}(\mathbb{R}^n) \oplus \mathcal{S}_{p^+<0}(\mathbb{R}^n)$.

(iii) Consider the following filtration of $\mathcal{S}(\mathbb{R}^n)$

$$\mathcal{S}(\mathbb{R}^n) \supset p^+\mathcal{S}(\mathbb{R}^n) \supset (p^+)^2\mathcal{S}(\mathbb{R}^n) \supset \cdots$$

Then $\mathcal{S}_{p^+}(\mathbb{R}^n) = \bigcap_{k\geq 0}(p^+)^k\mathcal{S}(\mathbb{R}^n)$, or, using categorical language,

$$\mathcal{S}_{p^+}(\mathbb{R}^n) = \varprojlim_{k\in\mathbb{N}} (p^+)^k\mathcal{S}(\mathbb{R}^n)$$

in the category of topological vector spaces.

(iv) If $Q \in \mathbb{C}[p^+, p_\perp]_{p^+}$, $g \in \mathcal{S}_{p^+}(\mathbb{R}^n)$, and α is a multi-index, then

$$f \mapsto Qf, \qquad f \mapsto gf, \qquad f \mapsto \partial^\alpha f$$

are continuous, linear mappings from $\mathcal{S}_{p^+}(\mathbb{R}^n)$ to $\mathcal{S}_{p^+}(\mathbb{R}^n)$. [1]

Proof. cf. [16]. □

[1] We consider Qf canonically as a function on $\mathbb{R}^n$ (cf. Remark 4.3)

In the following we identify $\mathcal{S}'_{p^+\gtrless 0}(\mathbb{R}^n)$ with the subspace $\{u \in \mathcal{S}'_{p^+}(\mathbb{R}^n) : u|_{p^+\lessgtr 0} \equiv 0\}$. Furthermore, we have canonical inclusion mappings $\mathcal{S}_{p^+\gtrless 0}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'_{p^+\gtrless 0}(\mathbb{R}^n)$ and $\mathcal{S}_{p^+}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'_{p^+}(\mathbb{R}^n)$.

Theorem 4.7. *(i)* $\mathcal{S}'_{p^+}(\mathbb{R}^n) = \mathcal{S}'_{p^+>0}(\mathbb{R}^n) \oplus \mathcal{S}'_{p^+<0}(\mathbb{R}^n)$.

(ii) For each $u \in \mathcal{S}'_{p^+\gtrless 0}(\mathbb{R}^n)$*, there is a sequence* $(u_i)_{i\in\mathbb{N}}$*,* $u_i \in \mathcal{S}_{p^+\gtrless 0}(\mathbb{R}^n)$*, converging to* u *in* $\mathcal{S}'_{\gtrless 0}(\mathbb{R}^n)$*. The same holds, if we replace* $\mathcal{S}_{p^+\gtrless 0}(\mathbb{R}^n)$ *(respectively* $\mathcal{S}'_{p^+\gtrless 0}(\mathbb{R}^n)$*) by* $\mathcal{S}_{p^+}(\mathbb{R}^n)$ *(respectively* $\mathcal{S}'_{p^+}(\mathbb{R}^n)$*).*

(iii) The isomorphisms $\nu^*_{\gtrless 0} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}_{p^+\gtrless 0}(\mathbb{R}^n)$ *extend (uniquely) to linear, sequentially continuous, bijective mappings* $\nu^*_{\gtrless 0} : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'_{p^+\gtrless 0}(\mathbb{R}^n)$*, where the inverse mappings are also sequentially continuous.*

Proof. cf. [16]. □

Definition 4.8. (a) We denote by $\mathcal{S}_{\partial_{x^-}}(\mathbb{R}^n)$ (or briefly $\mathcal{S}_{\partial_-}(\mathbb{R}^n)$) the complex vector space

$$\mathcal{S}_{\partial_{x^-}}(\mathbb{R}^n) := \bigcap_{m\geq 0} \partial^m_{x^-}\mathcal{S}(\mathbb{R}^n) = \{g : \forall m \geq 0\ \exists h \in \mathcal{S}(\mathbb{R}^n)\ g = \partial^m_{x^-} h\}$$

endowed with the subspace topology induced by $\mathcal{S}(\mathbb{R}^n)$ which makes $\mathcal{S}_{\partial_{x^-}}(\mathbb{R}^n)$ into a locally convex topological vector space. The dual space $\mathcal{S}'_{\partial_-}(\mathbb{R}^n)$ is called the space of *tame generalized functions*.

(b) The (continuous) inclusion mapping $\iota : \mathcal{S}_{\partial_-}(\mathbb{R}^n) \hookrightarrow \mathcal{S}(\mathbb{R}^n)$ induces a canonical (pull-back) mapping $\iota^* : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'_{\partial_-}(\mathbb{R}^n)$, $u \mapsto u^* := \iota^* u$; we call u^* the *relaxation* of u.

Notice that, by the Hahn-Banach Theorem, the mapping $u \mapsto u^*$ is surjective. The elements of the fibre over u^* are the regularizations of u^*.

Proposition 4.9. *The mapping* $\mathcal{F}^{\tilde{\mathbf{x}}\to\tilde{\mathbf{p}}}_{\mathbb{L}}$ *(cf. (2.1)) is a linear homeomorphism from* $\mathcal{S}(\mathbb{R}^n)$ *onto* $\mathcal{S}(\mathbb{R}^n)$ *which maps the subspace* $\mathcal{S}_{\partial_-}(\mathbb{R}^n)$ *onto the subspace* $\mathcal{S}_{p^+}(\mathbb{R}^n)$*. The canonical extension of* $\mathcal{F}^{\tilde{\mathbf{x}}\to\tilde{\mathbf{p}}}_{\mathbb{L}}$ *to* $\mathcal{S}'(\mathbb{R}^n)$*, also denoted by* $\mathcal{F}^{\tilde{\mathbf{x}}\to\tilde{\mathbf{p}}}_{\mathbb{L}}$*, maps* $\mathcal{S}'_{\partial_-}(\mathbb{R}^n)$ *onto* $\mathcal{S}'_{p^+}(\mathbb{R}^n)$*.*

Corollary 4.10. *The space* $\mathcal{S}_{\partial_-}(\mathbb{R}^n)$ *is a closed subspace of* $\mathcal{S}(\mathbb{R}^n)$*, hence* $\mathcal{S}_{\partial_-}(\mathbb{R}^n)$ *is a Fréchet space.*

Definition 4.11. A function $M \in C^\infty(\mathbb{R}^n\backslash\{p^+ = 0\}, \mathbb{C})$ is called a *multiplicator in* $\mathcal{S}_{p^+}(\mathbb{R}^n)$ if $Mf \in \mathcal{S}_{p^+}(\mathbb{R}^n)$ for every $f \in \mathcal{S}_{p^+}(\mathbb{R}^n)$.

Remark 4.12. As one can easily see, if M, M_1, M_2 are multiplicators in $\mathcal{S}_{p^+}(\mathbb{R}^n)$, $c_1, c_2 \in \mathbb{C}$ and α a multi-index, then also M_1M_2, $c_1M_1 + c_2M_2$ and $\partial^\alpha M$ are multiplicators in $\mathcal{S}_{p^+}(\mathbb{R}^n)$. Furthermore, from Theorem 4.6 it follows that any $Q \in C[p^+, p_\perp]_{p^+}$ and any $g \in \mathcal{S}_{p^+}(\mathbb{R}^n)$ is a multiplicator in $\mathcal{S}_{p^+}(\mathbb{R}^n)$.

By standard arguments, as in the case of generalized functions, we obtain:

Proposition 4.13. *A function* $M \in C^\infty(\mathbb{R}^n \setminus \{p^+ = 0\}, \mathbb{C})$ *is a multiplicator in* $\mathcal{S}_{p^+}(\mathbb{R}^n)$ *if and only if there exists for each multi-index* α *a positive constant* C_α *and a natural number* N_α *such that*

$$|\partial^\alpha M(\tilde{\mathbf{p}})| \leq C_\alpha \left(\frac{1 + |\tilde{\mathbf{p}}|}{|p^+|}\right)^{N_\alpha} \qquad \textit{for all multi-indices } \alpha \textit{ and } \tilde{\mathbf{p}} \in \mathbb{R}^n.$$

Corollary 4.14. *The functions $\Theta(\pm p^+)$, $|p^+|^k$ ($k \in \mathbb{Z}$) are multiplicators in $\mathcal{S}_{p^+}(\mathbb{R}^n)$, (hereby Θ is the Heaviside function defined by $\Theta(p^+) = 1$, if $p^+ > 0$, and $\Theta(p^+) = 0$, if $p^+ < 0$.)*

Definition 4.15. Given a multiplicator M in $\mathcal{S}_{p^+}(\mathbb{R}^n)$ and a squeezed generalized function $u \in \mathcal{S}'_{p^+}(\mathbb{R}^n)$, we define the squeezed generalized function $Mu \in \mathcal{S}'_{p^+}(\mathbb{R}^n)$ by $(Mu, f) = (u, Mf)$ for all $f \in \mathcal{S}_{p^+}(\mathbb{R}^n)$.

Our definition of a multiplicator in $\mathcal{S}_{p^+}(\mathbb{R}^n)$ is closely related to that of a multiplicator in $\mathcal{S}(\mathbb{R}^n)$ as defined in [2]. However, the set of multiplicators in $\mathcal{S}_{p^+}(\mathbb{R}^n)$ involves much more singular functions such as $1/p^+$ which is even non-locally integrable. This feature is crucial in solving the characteristic Cauchy problem.

4.2. **Solutions of the LCKG-equation.** In Section 3 we determined the general solution of the (homogeneous) KG-equation $(\Box + m^2)u = 0$ by determining the general solution of the associated division problem $(p^2 - m^2)v = 0$. Thereby we introduced the generalized functions $a_\pm(\mathbf{p})\delta(p^2 - m^2) \in \mathcal{S}'(\mathbb{R}^{1+n})$ where $a_\pm(\mathbf{p}) \in \mathcal{S}'(\mathbb{R}^n)$.

In this subsection we would like to determine the general solution of the (homogeneous) LCKG-equation $(\widetilde{\Box} + m^2)\tilde{u} = 0$ in the same way. The associated division problem reads now $(\tilde{p}^2 - m^2)\tilde{v} = 0$ (recall that $\tilde{p}^2 = 2p^+p^- - p_\perp^2$). We start by defining an important class of generalized functions on $\mathbb{R}^{1+n}$.

Definition 4.16. For each squeezed generalized function $b(\tilde{\mathbf{p}}) \in \mathcal{S}'_{p^+}(\mathbb{R}^n)$ we define the generalized functions $b(\tilde{\mathbf{p}})\delta_\pm(\tilde{p}^2 - m^2)$, $b(\tilde{\mathbf{p}})\delta(\tilde{p}^2 - m^2) \in \mathcal{S}'(\mathbb{R}^{1+n})$ by

$$(b(\tilde{\mathbf{p}})\delta_\pm(\tilde{p}^2 - m^2), f(\tilde{p})) = (b(\tilde{\mathbf{p}}), \frac{\Theta(\pm p^+)}{2|p^+|} f(\widetilde{\Omega}(\tilde{\mathbf{p}}))) \qquad (f \in \mathcal{S}(\mathbb{R}^{1+n})),$$

$$(b(\tilde{\mathbf{p}})\delta(\tilde{p}^2 - m^2), f(\tilde{p})) = (b(\tilde{\mathbf{p}}), \frac{1}{2|p^+|} f(\widetilde{\Omega}(\tilde{\mathbf{p}}))) \qquad (f \in \mathcal{S}(\mathbb{R}^{1+n})),$$

where $\widetilde{\Omega}(\tilde{\mathbf{p}}) = (\tilde{\mathbf{p}}, \tilde{\omega}(\tilde{\mathbf{p}}))$ and $\tilde{\omega}(\tilde{\mathbf{p}}) = \frac{p_\perp^2 + m^2}{2p^+}$, $p_\perp^2 = \sum_{i=1}^{n-1}(p^i)^2$.

Remark 4.17. The above definitions make sense since $\Theta(\pm p^+)$ and $|p^+|^{-1}$ are multiplicators in $\mathcal{S}_{p^+}(\mathbb{R}^n)$, by Corollary 4.14, and since $f \mapsto f(\tilde{\mathbf{p}}, \tilde{\omega}(\tilde{\mathbf{p}}))$ is a $\mathbb{C}$-linear, continuous mapping from $\mathcal{S}(\mathbb{R}^{1+n})$ to $\mathcal{S}_{p^+}(\mathbb{R}^n)$. This follows easily from $\widetilde{\Omega}_\pm = \kappa \circ \Omega_\pm \circ \nu_{\gtrless 0}$ and the fact that $f \mapsto f \circ \kappa \circ \Omega_\pm$ is a linear, continuous mapping from $\mathcal{S}(\mathbb{R}^{1+n})$ to $\mathcal{S}(\mathbb{R}^n)$, and $g \mapsto g \circ \nu_{\gtrless 0}$ is a linear, continuous mapping from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}_{p^+}(\mathbb{R}^n)$ (cf. Proposition 4.5).

Remark 4.18. a) If $b(\tilde{\mathbf{p}}) \in \mathcal{S}'_{p^+}(\mathbb{R}^n)$, $b_+(\tilde{\mathbf{p}}) \in \mathcal{S}'_{p^+>0}(\mathbb{R}^n)$ and $b_-(\tilde{\mathbf{p}}) \in \mathcal{S}'_{p^+<0}(\mathbb{R}^n)$ are given such that $b = b_+ + b_-$ holds in $\mathcal{S}'_{p^+}(\mathbb{R}^n)$ then

$$b(\tilde{\mathbf{p}})\delta(\tilde{p}^2 - m^2) = b_+(\tilde{\mathbf{p}})\delta_+(\tilde{p}^2 - m^2) + b_-(\tilde{\mathbf{p}})\delta_-(\tilde{p}^2 - m^2).$$

b) Let $\widetilde{\Gamma}_m = \{\tilde{p} \in \mathbb{R}^{1+n} : \tilde{p}^2 - m^2 = 0\}$ be the LC-mass hyperboloid, and $\widetilde{\Gamma}_m^\pm = \widetilde{\Gamma}_m \cap \{\pm p^- > 0\}$. Then, if $\kappa : \mathbb{M}^{1+n} \to \mathbb{L}^{1+n}$ is the transformation to LC-coordinates, $\kappa(\Gamma_m^\pm) = \widetilde{\Gamma}_m^\pm$. In contrast to the Minkowski-case, cf. Remark 3.2 a), the projection map $\widetilde{\Gamma}_m \to \mathbb{R}^n$, $\tilde{p} = (\tilde{\mathbf{p}}, p^-) \mapsto \tilde{\mathbf{p}}$ induces a homeomorphism from $\widetilde{\Gamma}_m$ onto $\mathbb{R}^n \setminus \{p^+ = 0\}$ whose inverse mapping is $\mathbb{R}^n \setminus \{p^+ = 0\} \to \widetilde{\Gamma}_m$, $\tilde{\mathbf{p}} \mapsto \widetilde{\Omega}(\tilde{\mathbf{p}})$. We denote by $\widetilde{\Omega}_\pm$ the restriction of $\widetilde{\Omega}$ to $\{\pm p^+ > 0\}$, then $\widetilde{\Omega}_\pm$ maps $\{\pm p^+ > 0\}$ onto $\widetilde{\Gamma}_m^\pm$.

c) If $b(\tilde{\mathbf{p}}) \subset \mathcal{S}_{p^+}(\mathbb{R}^n)(\subset \mathcal{S}'_{p^+}(\mathbb{R}^n))$ then

$$(b(\tilde{\mathbf{p}})\delta_\pm(\tilde{p}^2 - m^2), f(\tilde{\mathbf{p}})) = \int_{\widetilde{\Gamma}_m^\pm} \frac{(b \circ \widetilde{\Omega}^{-1})(\tilde{p}) f(\tilde{\mathbf{p}})}{|\nabla \widetilde{Q}(\tilde{p})|} d\tilde{S}(\tilde{p}),$$

where $\widetilde{Q}(\tilde{p}) = \tilde{p}^2 - m^2$ and $d\tilde{S}$ is the canonical surface measure on $\widetilde{\Gamma}_m$.

d) The support of $b(\tilde{\mathbf{p}})\delta_\pm(\tilde{p}^2 - m^2)$ is contained in $\widetilde{\Gamma}_m^\pm$.

Lemma 4.19. *The mappings $\mathcal{S}'_{p^+}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^{1+n})$, $b(\tilde{\mathbf{p}}) \mapsto b(\tilde{\mathbf{p}})\delta_\pm(\tilde{p}^2 - m^2)$ and $b(\tilde{\mathbf{p}}) \mapsto b(\tilde{\mathbf{p}})\delta(\tilde{p}^2 - m^2)$ are $\mathbb{C}$-linear and sequentially continuous.*

Proof. This follows immediately from the definitions of the generalized functions $b(\tilde{\mathbf{p}})\delta_\pm(\tilde{p}^2 - m^2)$ and $b(\tilde{\mathbf{p}})\delta(\tilde{p}^2 - m^2)$. □

The next proposition is essential for the sequel since it gives us the proper transformation law between the generalized functions $a(\mathbf{p})\delta_\pm(p^2 - m^2)$ ($a(\mathbf{p}) \in \mathcal{S}'(\mathbb{R}^n)$) and $b(\tilde{\mathbf{p}})\delta_\pm(\tilde{\mathbf{p}}^2 - m^2)$ ($b(\tilde{\mathbf{p}}) \in \mathcal{S}'_{p^+}(\mathbb{R}^n)$). Notice that $\nu^*_{\gtrless 0}$ is an isomorphism from $\mathcal{S}'(\mathbb{R}^n)$ onto $\mathcal{S}'_{p^+ \gtrless 0}(\mathbb{R}^n)$.

Proposition 4.20. *Let $a(\mathbf{p}) \in \mathcal{S}'(\mathbb{R}^n)$ be a generalized function and let $\kappa : \mathbb{R}^{1+n} \to \mathbb{R}^{1+n}$ be the transformation to LC-coordinates. Then*

$$a(\mathbf{p})\delta_\pm(p^2 - m^2) \circ \kappa^{-1} = (\nu^*_{\gtrless 0} a)(\tilde{\mathbf{p}})\delta_\pm(\tilde{p}^2 - m^2).$$

Proof. Both sides of the above equation depend (sequentially) continuously on $a(\mathbf{p}) \in \mathcal{S}'(\mathbb{R}^n)$. Hence it is enough to show equality, if $a(\mathbf{p})$ is from the dense subspace $\mathcal{S}(\mathbb{R}^n)$ of $\mathcal{S}'(\mathbb{R}^n)$. So, assume $a(\mathbf{p}) \in \mathcal{S}(\mathbb{R}^n)$ and let $f \in \mathcal{S}(\mathbb{R}^{1+n})$. Then

$$\begin{aligned}
(a(\mathbf{p})\delta_\pm(p^2 - m^2) \circ \kappa^{-1}, f) &= \int \frac{d^n\mathbf{p}}{2\omega(\mathbf{p})} a(\mathbf{p})(f \circ \kappa \circ \Omega_\pm)(\mathbf{p}) = \\
&= \int_{\Gamma_m^\pm} \frac{(a \circ \Omega_\pm^{-1})(f \circ \kappa)}{|\nabla Q|} dS = \int_{\widetilde{\Gamma}_m^\pm} \frac{(a \circ \Omega_\pm^{-1} \circ \kappa^{-1}) f}{|\nabla \widetilde{Q}|} d\tilde{S} = \\
&= \int_{p^+ \gtrless 0} \frac{d^n\tilde{\mathbf{p}}}{2|p^+|} (a \circ \Omega_\pm^{-1} \circ \kappa^{-1} \circ \widetilde{\Omega}_\pm)(\tilde{\mathbf{p}})(f \circ \widetilde{\Omega}_\pm)(\tilde{\mathbf{p}}) = \\
&= \left((\nu^*_{\gtrless 0} a)(\tilde{\mathbf{p}})\delta_\pm(\tilde{p}^2 - m^2), f\right).
\end{aligned}$$

□

Corollary 4.21. *Let $b_\pm(\tilde{\mathbf{p}}) \in \mathcal{S}'_{p^+ \gtrless 0}(\mathbb{R}^n)$ and $b(\tilde{\mathbf{p}}) \in \mathcal{S}'_{p^+}(\mathbb{R}^n)$.*

(i) $b_\pm(\tilde{\mathbf{p}})\delta_\pm(\tilde{p}^2 - m^2) = 0$ if and only if $b_\pm(\tilde{\mathbf{p}}) = 0$.

(ii) $b(\tilde{\mathbf{p}})\delta(\tilde{p}^2 - m^2) = 0$ if and only if $b(\tilde{\mathbf{p}}) = 0$.

Proof. (i) This can be proven directly as in Lemma 3.4(i), however, we will reduce it to Lemma 3.4. Assume $\tilde{u} = b_\pm(\tilde{\mathbf{p}})\delta_\pm(\tilde{p}^2 - m^2) = 0$, then also $u = \tilde{u} \circ \kappa = 0$. By Proposition 4.20, $u = a_\pm(\mathbf{p})\delta_\pm(p^2 - m^2)$ with $a_\pm = (\nu^*_{\gtrless 0})^{-1} b_\pm \in \mathcal{S}'(\mathbb{R}^n)$. Now, by Lemma 3.4 (i), $a_\pm(\mathbf{p}) = 0$ and hence $b_\pm(\tilde{\mathbf{p}}) = 0$.

(ii) Assume $b(\tilde{\mathbf{p}})\delta(\tilde{p}^2 - m^2) = 0$. By Theorem 4.7 (i) there are unique $b_\pm(\tilde{\mathbf{p}}) \in \mathcal{S}'_{p^+ \gtrless 0}(\mathbb{R}^n)$ such that $b(\tilde{\mathbf{p}}) = b_+(\tilde{\mathbf{p}}) + b_-(\tilde{\mathbf{p}})$. Since from $0 = b(\tilde{\mathbf{p}})\delta(p^2 - m^2) = b_+(\tilde{\mathbf{p}})\delta_+(\tilde{p}^2 - m^2) + b_-(\tilde{\mathbf{p}})\delta_-(\tilde{p}^2 - m^2)$ we obtain $b_+(\tilde{\mathbf{p}})\delta_+(\tilde{p}^2 - m^2) = -b_-(\tilde{\mathbf{p}})\delta_-(\tilde{p}^2 - m^2)$ and since these distributions have disjoint supports, the assertion follows from (i). □

Remark 4.22. Corollary 4.21 shows that $\mathcal{S}'_{p^+\gtrless 0}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^{1+n})$, $b(\tilde{\mathbf{p}}) \mapsto b(\tilde{\mathbf{p}})\delta_\pm(\tilde{p}^2 - m^2)$ and $\mathcal{S}'_{p^+}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^{1+n})$, $b(\tilde{\mathbf{p}}) \mapsto b(\tilde{\mathbf{p}})\delta(\tilde{p}^2 - m^2)$ are injective maps.

Corollary 4.23. *The general solution of the division problem* $(\tilde{p}^2 - m^2)\tilde{u} = 0$ *in* $\mathcal{S}'(\mathbb{R}^{1+n})$ *is given by*

$$\tilde{u} = b_+(\tilde{\mathbf{p}})\delta_+(\tilde{p}^2 - m^2) + b_-(\tilde{\mathbf{p}})\delta_-(\tilde{p}^2 - m^2) \tag{4.2}$$

with $b_+(\tilde{\mathbf{p}}) \in \mathcal{S}'_{p^+>0}(\mathbb{R}^n)$ *and* $b_-(\tilde{\mathbf{p}}) \in \mathcal{S}'_{p^+<0}(\mathbb{R}^n)$*, or, equivalently, by*

$$\tilde{u} = b(\tilde{\mathbf{p}})\delta(\tilde{p}^2 - m^2)$$

with $b \in \mathcal{S}'_{p^+}(\mathbb{R}^n)$.

Proof. The mapping $u \mapsto \tilde{u} = u \circ \kappa^{-1}$ defines a 1-1 correspondence between the solutions of $(p^2 - m^2)u = 0$ and the solutions of $(\tilde{p}^2 - m^2)\tilde{u} = 0$. The general solution of $(p^2 - m^2)u = 0$ is of the form $u = a_+(\mathbf{p})\delta_+(p^2 - m^2) + a_-(\mathbf{p})\delta_-(p^2 - m^2)$ with $a_\pm(\mathbf{p}) \in \mathcal{S}'(\mathbb{R}^n)$, hence the assertion follows from Proposition 4.20. ☐

Remark 4.24. The general solution of the division problem $(\tilde{p}^2 - m^2)\tilde{u} = 1$ in $\mathcal{S}'(\mathbb{R}^{1+n})$ is given by

$$\tilde{u} = \mathrm{PV}\frac{1}{\tilde{p}^2 - m^2} + b_+(\tilde{\mathbf{p}})\delta_+(\tilde{p}^2 - m^2) + b_-(\tilde{\mathbf{p}})\delta_-(\tilde{p}^2 - m^2)$$

where $b_\pm(\tilde{\mathbf{p}}) \in \mathcal{S}'_{p^+\gtrless 0}(\mathbb{R}^n)$, or, equivalently, by

$$\tilde{u} = \mathrm{PV}\frac{1}{\tilde{p}^2 - m^2} + b(\tilde{\mathbf{p}})\delta(\tilde{p}^2 - m^2)$$

where $b(\tilde{\mathbf{p}}) \in \mathcal{S}'_{p^+}(\mathbb{R}^n)$ where PV denotes the principal value.

Corollary 4.25. *Let* $u \in \mathcal{S}'(\mathbb{R}^{1+n})$ *be a solution of the LCKG-equation* $(\widetilde{\Box} + m^2)u = 0$. *Then there is a unique* $b(\tilde{\mathbf{p}}) \in \mathcal{S}'_{p^+}(\mathbb{R}^n)$ *such that* $u = \mathcal{F}_{\mathbb{L}}^{-1}(b(\tilde{\mathbf{p}})\delta(\tilde{p}^2 - m^2))$.

Proof. Since $u \in \mathcal{S}'(\mathbb{R}^{1+n})$ is a solution of the LCKG-equation, $\mathcal{F}_{\mathbb{L}}(u) = u^{\wedge\mathbb{L}}$ is a solution of the division problem $(\tilde{p}^2 - m^2)u^{\wedge\mathbb{L}} = 0$. The existence follows now from Corollary 4.23 and the uniqueness from Corollary 4.21. ☐

4.3. **The tame restriction of a generalized function.** There are several equivalent ways to define canonically the restriction of a distribution u on $\mathbb{R}^{1+n}$ to a hyperplane $\Sigma \subset \mathbb{R}^{1+n}$. One approach uses an extended construction of the pullback of a distribution to define the restriction $u|_\Sigma$ as the pullback $\iota^* u$, where $\iota : \Sigma \hookrightarrow \mathbb{R}^{1+n}$ is the inclusion mapping. By this, the restriction is canonically definable if and only if the normal bundle of (the submanifold) Σ doesn't intersect the wave front set of u (cf. [11], 8.2). By another way one uses the definition of dependence of a distribution on a parameter and defines the restriction by just fixing the parameter (cf., e.g., [2] 2.6.). We will define the tame restriction of a generalized function by using the second method, where connections with pullbacks and wave front sets will be elaborated in a further paper [17].

Definition 4.26. A tame generalized function $u = u(x^+, \tilde{\mathbf{x}}) \in \mathcal{S}'_{\partial_-}(\mathbb{R}^{1+n})$ is called C^∞-*dependent* on $x^+ \in \Omega$ as a parameter ($\Omega \subset \mathbb{R}$ open) if there exists a family $(u_{x^+})_{x^+ \in \Omega}$ in $\mathcal{S}'_{\partial_-}(\mathbb{R}^n)$, $u_{x^+} = u_{x^+}(\tilde{\mathbf{x}})$, such that

$$(u(x^+, \tilde{\mathbf{x}}), f(x^+) \otimes g(\tilde{\mathbf{x}})) = \int_\Omega dx^+ (u_{x^+}, g) f(x^+)$$

for all $f \in \mathcal{D}(\Omega)$, $g \in \mathcal{S}_{\partial_-}(\mathbb{R}^n)$, and such that

$$\Omega \to \mathbb{C},\ x^+ \mapsto (u_{x^+}, g)$$

is C^∞ for all $g \in \mathcal{S}_{\partial_-}(\mathbb{R}^n)$.

Remark 4.27. Obviously, the family $(u_{x^+})_{x^+ \in \Omega}$ is uniquely determined by u.

Since the LCKG-operator $\widetilde{\Box} + m^2$ is not hypoelliptic with respect to x^+ there are solutions $u \in \mathcal{D}'(\mathbb{R}^{1+n})$ of the LCKG-equation which are not C^∞-dependent on x^+ as a parameter. For instance, the positive/negative frequency Pauli-Jordan functions $D^{(\pm)} = \frac{\pm 1}{i(2\pi)^3} \mathcal{F}_{\mathbb{M}}(\delta_\pm(p^2 - m^2)) \in \mathcal{S}'(\mathbb{R}^4)$ do not have a canonical restriction to $\{x^0 + x^3 = 0\}$. This can be seen by considering the wave front set of $D^{(\pm)}$ (see [19]). Now the following proposition shows that we can retrieve parameter dependence if we consider the relaxation u^* instead of u.

Proposition 4.28. *Assume $b(\tilde{\mathbf{p}}) \in \mathcal{S}'_{p^+}(\mathbb{R}^n)$ and let $u(\tilde{x}) = \mathcal{F}_{\mathbb{L}}^{-1}(b(\tilde{\mathbf{p}})\delta(\tilde{p}^2 - m^2)) \in \mathcal{S}'(\mathbb{R}^{1+n})$. Then $u^*(\tilde{x}) = u^*(x^+, \tilde{\mathbf{x}})$ is C^∞-dependent on x^+ as a parameter, and for all $x^+ \in \mathbb{R}$*

$$u^*_{x^+} = \frac{1}{4\pi} (\mathcal{F}_{\mathbb{L}}^{\tilde{\mathbf{x}} \to \tilde{\mathbf{p}}})^{-1} \left(\frac{e^{-i\tilde{\omega}(\tilde{\mathbf{p}})x^+}}{|p^+|} b(\tilde{\mathbf{p}}) \right).$$

Proof. For each $x^+ \in \mathbb{R}$ we set $u^*_{x^+} = \frac{1}{4\pi} (\mathcal{F}_{\mathbb{L}}^{\tilde{\mathbf{x}} \to \tilde{\mathbf{p}}})^{-1} \left(\frac{e^{-i\tilde{\omega}(\tilde{\mathbf{p}})x^+}}{2|p^+|} b(\tilde{\mathbf{p}}) \right)$. Since, by Proposition 4.13, $|p^+|^{-1} e^{-i\tilde{\omega}(\tilde{\mathbf{p}})x^+}$ is a multiplicator in $\mathcal{S}_{p^+}(\mathbb{R}^n)$, $u_{x^+} \in \mathcal{S}'_{\partial_-}(\mathbb{R}^n)$ is well-defined for each $x^+ \in \mathbb{R}$. Moreover, for every $g \in \mathcal{S}_{p^+}(\mathbb{R}^n)$,

$$\mathbb{R} \to \mathbb{C},\ x^+ \mapsto (b(\tilde{\mathbf{p}}), 2^{-1}|p^+|^{-1} e^{-i\tilde{\omega}(\tilde{\mathbf{p}})x^+} g(\tilde{\mathbf{p}}))$$

is a C^∞-mapping and thus $\mathbb{R} \to \mathbb{C}$, $x^+ \mapsto (u_{x^+}, g)$ is a C^∞-mapping, too. It remains to show, that

$$(u(x^+, \tilde{\mathbf{x}}), f(x^+) g(\tilde{\mathbf{x}})) = \int (u_{x^+}, g) f(x^+) dx^+ \tag{4.3}$$

for each $f(x^+) \in \mathcal{S}(\mathbb{R})$ and $g(\tilde{\mathbf{x}}) \in \mathcal{S}_{\partial_-}(\mathbb{R}^n)$.

Case 1: Firstly, we assume that $b(\tilde{\mathbf{p}}) \in \mathcal{S}_{p^+}(\mathbb{R}^n)(\subset \mathcal{S}'_{p^+}(\mathbb{R}^n))$. By an easy computation we obtain

$$\begin{aligned}
(u^*, f\otimes g) &= (b(\tilde{\mathbf{p}})\delta(\tilde{p}^2 - m^2), \mathcal{F}_{\mathbb{L}}^{-1}(f\otimes g)) \\
&= \frac{1}{(2\pi)^{n+1}} \int dx^+ f(x^+) \int d^n\tilde{\mathbf{x}}\, g(\tilde{\mathbf{x}}) \int \frac{d^n\tilde{\mathbf{p}}}{2|p^+|} b(\tilde{\mathbf{p}}) e^{-i(\tilde{\omega}(\tilde{\mathbf{p}})x^+ + p^+x^- - p_\perp\cdot x_\perp)} \\
&= \frac{1}{2\pi} \int dx^+ f(x^+) \int \frac{d^n\tilde{\mathbf{p}}}{2|p^+|} b(\tilde{\mathbf{p}}) e^{-i\tilde{\omega}(\tilde{\mathbf{p}})x^+}((\mathcal{F}_{\mathbb{L}}^{\tilde{\mathbf{x}}\to\tilde{\mathbf{p}}})^{-1}g)(\tilde{\mathbf{p}}) \\
&= \int dx^+ (u^*_{x^+}, g) f(x^+)
\end{aligned}$$

for $f \in \mathcal{S}(\mathbb{R})$, $g \in \mathcal{S}_{\partial_-}(\mathbb{R}^n)$.

Case 2: Now we consider the general case $b(\tilde{\mathbf{p}}) \in \mathcal{S}'_{p^+}(\mathbb{R}^n)$. By Theorem 4.7 there is a sequence $(b^{(m)}(\tilde{\mathbf{p}}))$ in $\mathcal{S}_{p^+}(\mathbb{R}^n)$ converging to $b(\tilde{\mathbf{p}})$ in $\mathcal{S}'_{p^+}(\mathbb{R}^n)$. Define $u^{(m)} \in \mathcal{S}'(\mathbb{R}^{1+n})$ by $(u^{(m)})^{\wedge\mathbb{L}} = b^{(m)}(\tilde{\mathbf{p}})\delta(\tilde{p}^2 - m^2)$. By Lemma 4.19 the sequence $(u^{(m)})$ converges to u in $\mathcal{S}'(\mathbb{R}^{1+n})$ and, by construction, $(u^{(m)*}_{x^+})$ converges to $u^*_{x^+}$ in $\mathcal{S}'_{\partial_-}(\mathbb{R}^n)$. By the first case we have

$$(u^*(x^+, \tilde{\mathbf{x}}), f(x^+)g(\tilde{\mathbf{x}})) = \lim_{m\to\infty} \int (u^{(m)*}_{x^+}, g) f(x^+) dx^+ \tag{4.4}$$

for $f \in \mathcal{S}(\mathbb{R})$, $g \in \mathcal{S}_{\partial_-}(\mathbb{R}^n)$. Since $|(u^{(m)*}_{x^+}, g)| \le \int \frac{d^n\tilde{\mathbf{p}}}{2|p^+|} |b^{(m)}(\tilde{\mathbf{p}})h(\tilde{\mathbf{p}})| = C < \infty$ ($h = (\mathcal{F}_{\mathbb{L}}^{\tilde{\mathbf{x}}\to\tilde{\mathbf{p}}})^{-1}g$), where the constant C does not depend on x^+, the right-hand side of (4.4) equals $\int (u_{x^+}, g) f(x^+) dx^+$, hence we have shown (4.3). ☐

Remark 4.29. To see in the above proof that, for each $g \in \mathcal{S}_{p^+}(\mathbb{R}^n)$, the mapping $\mathbb{R} \to \mathbb{C}$, $x^+ \mapsto (b(\tilde{\mathbf{p}}), 2^{-1}|p^+|^{-1}e^{-i\tilde{\omega}(\tilde{\mathbf{p}})x^+}g(\tilde{\mathbf{p}}))$ is C^∞, notice that for any multiplicator M in $\mathcal{S}_{p^+}(\mathbb{R}^n)$ the sequences

$$e^{-i\tilde{\omega}(\tilde{\mathbf{p}})(x^+ + h)} M(\tilde{\mathbf{p}})g(\tilde{\mathbf{p}}) \to e^{-i\tilde{\omega}(\tilde{\mathbf{p}})x^+} M(\tilde{\mathbf{p}})g(\tilde{\mathbf{p}}) \qquad (h \to 0)$$

and

$$\frac{1}{h}\left(e^{-i\tilde{\omega}(\tilde{\mathbf{p}})(x^+ + h)} - e^{-i\tilde{\omega}(\tilde{\mathbf{p}})x^+}\right) M(\tilde{\mathbf{p}})g(\tilde{\mathbf{p}}) \to -i\tilde{\omega}(\tilde{\mathbf{p}})e^{-i\tilde{\omega}(\tilde{\mathbf{p}})x^+} M(\tilde{\mathbf{p}})g(\tilde{\mathbf{p}}) \qquad (h \to 0)$$

converge in $\mathcal{S}_{p^+}(\mathbb{R}^n)$ for every $x^+ \in \mathbb{R}$.

Corollary 4.30. *The relaxation $u^*(x^+, \tilde{\mathbf{x}})$ of any solution $u(x^+, \tilde{\mathbf{x}}) \in \mathcal{S}'(\mathbb{R}^{1+n})$ of the LCKG-equation $(\widetilde{\Box} + m^2)u = 0$ is C^∞-dependent on $x^+ \in \mathbb{R}$ as a parameter.*

Proof. Let $u = u(x^+, \tilde{\mathbf{x}}) \in \mathcal{S}'(\mathbb{R}^{1+n})$ be a solution of the LCKG-equation. By Corollary 4.25 there is a (unique) $b(\tilde{\mathbf{p}}) \in \mathcal{S}'_{p^+}(\mathbb{R}^n)$ such that $u^{\wedge\mathbb{L}} = b(\tilde{\mathbf{p}})\delta(\tilde{p}^2 - m^2)$. Now apply Proposition 4.28. ☐

Definition 4.31. A generalized function $u(x^+, \tilde{\mathbf{p}}) \in \mathcal{S}'(\mathbb{R}^{1+n})$ admits a *tame restriction to* $\{x^+ = 0\}$ if there is an open neighbourhood Ω of $0 \in \mathbb{R}$ such that the relaxation $u^*(x^+, \tilde{\mathbf{x}})$ of u is C^∞-dependent on $x^+ \in \Omega$ as a parameter. If $(u^*_{x^+})_{x^+\in\Omega}$ is the corresponding family in $\mathcal{S}'_{\partial_-}(\mathbb{R}^n)$, we set $u|^*_{x^+=0} = u^*_0$ and call $u|^*_{x^+=0}$ the *tame restriction of u to* $\{x^+ = 0\}$.

Corollary 4.32. *Any solution $u \in \mathcal{S}'(\mathbb{R}^{1+n})$ of the LCKG-equation admits a tame restriction to $\{x^+ = 0\}$.*

5. The characteristic Cauchy problem

In the preceding section we have treated the problem of restricting solutions $u(\tilde{x}) \in \mathcal{S}'(\mathbb{R}^{1+n})$ of the LCKG-equation $(\widetilde{\Box} + m^2)u = 0$ to $\{x^+ = 0\}$. As already mentioned, there exist solutions $u \in \mathcal{S}'(\mathbb{R}^n)$ which do not have a canonical restriction to $\{x^+ = 0\}$ in $\mathcal{S}'(\mathbb{R}^n)$. We solved this problem by introducing the space $\mathcal{S}_{\partial_-}(\mathbb{R}^n)$ along with its dual space $\mathcal{S}'_{\partial_-}(\mathbb{R}^n)$. Since $\mathcal{S}_{\partial_-}(\mathbb{R}^n)$ is a subspace of $\mathcal{S}(\mathbb{R}^n)$, the elements of $\mathcal{S}'_{\partial_-}(\mathbb{R}^n)$ may have more singularities than those in $\mathcal{S}'(\mathbb{R}^n)$. Now any solution of the LCKG-equation in $\mathcal{S}'_{\partial_-}(\mathbb{R}^{1+n})$ is C^∞-dependent on x^+ as a parameter, and yet has a restriction to $\{x^+ = 0\}$ in $\mathcal{S}'_{\partial_-}(\mathbb{R}^n)$. Hence it is possible to assign to each solution $u \in \mathcal{S}'(\mathbb{R}^{1+n})$ of the LCKG-equation a tame restriction $u|^*_{x^+=0}$ to $\{x^+ = 0\}$ in $\mathcal{S}'_{\partial_-}(\mathbb{R}^n)$ by considering u canonically as an element u^* of $\mathcal{S}'_{\partial_-}(\mathbb{R}^{1+n})$ – we call u^* the relaxation of u. This makes possible to consider the following characteristic Cauchy problem of the LCKG-equation:

$$(\widetilde{\Box} + m^2)u = 0, \qquad u|^*_{x^+=0} = u_0, \qquad (\widetilde{\Box} = 2\partial_+\partial_- - \sum_{i=1}^{n-1} \partial_i^2) \tag{5.1}$$

where $u = \mathcal{S}'(\mathbb{R}^{1+n})$ and $u_0 \in \mathcal{S}'_{\partial_-}(\mathbb{R}^n)$.

5.1. **Existence and uniqueness.** Since $\mathcal{S}_{\partial_-}(\mathbb{R}^{1+n})$ is not dense in $\mathcal{S}(\mathbb{R}^{1+n})$, the induced (relaxation) map $\mathcal{S}'(\mathbb{R}^{1+n}) \to \mathcal{S}'_{\partial_-}(\mathbb{R}^{1+n})$, $u \mapsto u^*$ is by no means injective. However, we will show in this subsection that this is true on the subspace of all $u \in \mathcal{S}'(\mathbb{R}^{1+n})$ which are solutions of the LCKG-equation. Moreover, such u are uniquely determined by their tame restriction $u|^*_{x^+=0} \in \mathcal{S}'_{\partial_-}(\mathbb{R}^n)$. Furthermore, we will show that for any $u_0 \in \mathcal{S}'_{\partial_-}(\mathbb{R}^n)$ there exists a solution $u \in \mathcal{S}'(\mathbb{R}^{1+n})$ of the LCKG-equation such that $u|^*_{x^+=0} = u_0$.

Lemma 5.1. *Assume $u \in \mathcal{S}'(\mathbb{R}^{1+n})$ is a solution of the LCKG-equation $(\widetilde{\Box} + m^2)u = 0$ and let $u_0 = u|^*_{x^+=0} \in \mathcal{S}'_{\partial_-}(\mathbb{R}^n)$ be the tame restriction of u to $\{x^+ = 0\}$. Then*

$$u^{\wedge\mathbb{L}} = 4\pi|p^+|\mathcal{F}_{\mathbb{L}}^{\tilde{\mathbf{x}}\to\tilde{\mathbf{p}}}(u_0)\,\delta(\tilde{p}^2 - m^2). \tag{5.2}$$

Proof. By Corollary 4.25 there is a unique $b(\tilde{\mathbf{p}}) \in \mathcal{S}'_{p^+}(\mathbb{R}^n)$ such that $u^{\wedge\mathbb{L}} = b(\tilde{\mathbf{p}})\delta(\tilde{p}^2 - m^2)$, and by Proposition 4.28 we obtain $\mathcal{F}_{\mathbb{L}}^{\tilde{\mathbf{x}}\to\tilde{\mathbf{p}}}(u_0) = \frac{1}{4\pi|p^+|}b(\tilde{\mathbf{p}})$. □

Theorem 5.2. *(i) Suppose $u \in \mathcal{S}'(\mathbb{R}^{1+n})$ is a solution of the LCKG-equation and $u|^*_{x^+=0} = 0$ in $\mathcal{S}'_{\partial_-}(\mathbb{R}^n)$, then $u = 0$.*

(ii) For each $u_0 \in \mathcal{S}'_{\partial_-}(\mathbb{R}^n)$ there exists a (unique) solution $u \in \mathcal{S}'(\mathbb{R}^{1+n})$ of the LCKG-equation such that $u|^*_{x^+=0} = u_0$.

Proof. (i) Since u is a solution of the LCKG-equation, $u^{\wedge\mathbb{L}} = 4\pi|p^+|\mathcal{F}_{\mathbb{L}}^{\tilde{\mathbf{x}}\to\tilde{\mathbf{p}}}(u_0)\,\delta(\tilde{p}^2 - m^2)$, by Lemma 5.1, where $u_0 = u|^*_{x^+=0}$. By assumption $u_0 = 0$, hence $u = 0$.

(ii) Assume $u_0 \in \mathcal{S}'_{\partial_-}(\mathbb{R}^{n-1})$. Define $u \in \mathcal{S}'(\mathbb{R}^n)$ by $u^{\wedge\mathbb{L}} = 4\pi|p^+|\mathcal{F}_{\mathbb{L}}^{\tilde{\mathbf{x}}\to\tilde{\mathbf{p}}}(u_0)\,\delta(\tilde{p}^2 - m^2)$. Then $(\tilde{p}^2 - m^2)u^{\wedge\mathbb{L}} = 0$ and thus $(\widetilde{\Box} + m^2)u = 0$. By Lemma 5.1 and Corollary 4.25 we obtain $4\pi|p^+|\mathcal{F}_{\mathbb{L}}^{\tilde{\mathbf{x}}\to\tilde{\mathbf{p}}}(u_0) = 4\pi|p^+|\mathcal{F}_{\mathbb{L}}^{\tilde{\mathbf{x}}\to\tilde{\mathbf{p}}}(u|^*_{x^+=0})$, and thus $u|^*_{x^+=0} = u_0$ – notice that $1/|p^+|$ is a multiplicator in $\mathcal{S}'_{p^+}(\mathbb{R}^n)$. □

The uniqueness in Theorem 5.2 implies the following surprising result.

Corollary 5.3. *Let $v \in \mathcal{S}'(\mathbb{R}^{1+n})$ be an arbitrary generalized function. Then any solution $u \in \mathcal{S}'(\mathbb{R}^{1+n})$ of the inhomogeneous LCKG-equation*

$$(\widetilde{\square} + m^2)u = v \tag{5.3}$$

is uniquely determined by its relaxation $u^* \in \mathcal{S}'_{\partial_-}(\mathbb{R}^{1+n})$.

Proof. If $v = 0$ then the statement follows immediately from the uniqueness in Theorem 5.2. Now, in the general case, let $u_1, u_2 \in \mathcal{S}'(\mathbb{R}^{1+n})$ be two solutions of (5.3) such that $u_1^* = u_2^*$. Since $u_1 - u_2$ is a solution of the homogeneous LCKG-equation and $(u_1 - u_2)^* = u_1^* - u_2^* = 0$, we obtain $u_1 - u_2 = 0$ from the first case. $\square$

5.2. **Connection between the characteristic and the non-characteristic Cauchy problem.** Let $u \in \mathcal{S}'(\mathbb{R}^{1+n})$ be a solution of the Klein-Gordon equation; then surely $\tilde{u} = u \circ \kappa^{-1}$ solves the LC-Klein-Gordon equation. Hence we can consider the several initial data $u_0 = u|_{x^0=0}$, $u_1 = \partial_0 u|_{x^0=0} \in \mathcal{S}'(\mathbb{R}^n)$ and $\tilde{u}_0 = \tilde{u}|^*_{x^+=0} \in \mathcal{S}'_{\partial_-}(\mathbb{R}^n)$. Since we have explicit formulas relating the initial data u_0, u_1 to u (cf. Propositions 3.5, 3.6) and the initial data $\tilde{u}_0$ to $\tilde{u}$ (cf. (5.2)) we can make use of the transformation law of Proposition 4.20 to obtain a transformation law between these initial data. Especially, we obtain again a proof of existence and uniqueness of the characteristic Cauchy problem (5.1).

Theorem 5.4. *Suppose $u \in \mathcal{S}'(\mathbb{R}^{1+n})$ is a solution of the Klein-Gordon equation and let $\tilde{u} = u \circ \kappa^{-1}$. Then, between $u_0 = u|_{x^0=0}$, $u_1 = \partial_0 u|_{x^0=0}$ and $\tilde{u}_0 = \tilde{u}|^*_{x^+=0}$, the following transformation laws hold:*

$$\tilde{u}_0^{\sqcap} = \frac{1}{2|p^+|}\left(\nu^*_{>0}(\omega u_0^{\wedge} + i u_1^{\wedge}) + \nu^*_{<0}(\omega u_0^{\wedge} - i u_1^{\wedge})\right), \tag{5.4}$$

and

$$u_0^{\wedge} = \frac{1}{\omega}\left(\mu^*_{>0}(|p^+|\Theta(p^+)\tilde{u}_0^{\sqcap}) + \mu^*_{<0}(|p^+|\Theta(-p^+)\tilde{u}_0^{\sqcap})\right), \tag{5.5}$$

$$u_1^{\wedge} = \frac{1}{i}\left(\mu^*_{>0}(|p^+|\Theta(p^+)\tilde{u}_0^{\sqcap}) - \mu^*_{<0}(|p^+|\Theta(-p^+)\tilde{u}_0^{\sqcap})\right), \tag{5.6}$$

where $\mu_{\gtrless 0} = \nu^{-1}_{\gtrless 0} : \mathbb{R}^n \to \mathbb{R}^n \backslash \{\mp p^+ \geq 0\}$, $\mathbf{p} \mapsto \tilde{\mathbf{p}} = (p^+, p_\perp) = (\frac{1}{\sqrt{2}}(p^n \pm \omega(\mathbf{p})), p^1, \ldots, p^{n-1})$, and $\omega = \omega(\mathbf{p}) = \sqrt{\mathbf{p}^2 + m^2}$.

Proof. Let $u \in \mathcal{S}'(\mathbb{R}^{1+n})$ be a solution of the Klein-Gordon equation, and let $\tilde{u} = u \circ \kappa^{-1}$. Then, by Propositions 3.5, 3.6, $u^{\wedge\mathbb{M}} = a_0(\mathbf{p})\delta_+(p^2 - m^2) + a_1(\mathbf{p})\delta_-(p^2 - m^2)$, where $a_0 = 2\pi(\omega u_0^{\wedge} + i u_1^{\wedge}) \in \mathcal{S}'(\mathbb{R}^n)$ and $a_1 = 2\pi(\omega u_0^{\wedge} - i u_1^{\wedge}) \in \mathcal{S}'(\mathbb{R}^n)$. On the other hand, by Lemma 5.1, $u^{\wedge\mathbb{L}} = b(\tilde{\mathbf{p}})\delta(\tilde{p}^2 - m^2)$, where $b(\tilde{\mathbf{p}}) = 4\pi|p^+|\tilde{u}_0^{\sqcap}(\tilde{\mathbf{p}})$. We write $b(\tilde{\mathbf{p}})\delta(\tilde{p}^2 - m^2) = b_0(\tilde{\mathbf{p}})\delta_+(\tilde{p}^2 - m^2) + b_1(\tilde{\mathbf{p}})\delta_-(\tilde{p}^2 - m^2)$ with $b_0(\tilde{\mathbf{p}}) = \Theta(p^+)b(\tilde{\mathbf{p}})$ and $b_1(\tilde{\mathbf{p}}) = \Theta(-p^+)b(\tilde{\mathbf{p}})$. Since $\tilde{u}^{\wedge\mathbb{L}} = u^{\wedge\mathbb{M}} \circ \kappa^{-1}$ we obtain $b_0 = \nu^*_{>0}(a_0)$ and $b_1 = \nu^*_{<0}(a_1)$ by Proposition 4.20 and Corollary 4.21, and hence

$$2|p^+|\Theta(p^+)\tilde{u}_0^{\sqcap} = \nu^*_{>0}(\omega u_0^{\wedge} + i u_1^{\wedge}),$$
$$2|p^+|\Theta(-p^+)\tilde{u}_0^{\sqcap} = \nu^*_{<0}(\omega u_0^{\wedge} - i u_1^{\wedge}),$$

from which the transformation laws easily follow. $\square$

Corollary 5.5. *Suppose the initial data* $u_0, u_1 \in \mathcal{S}'(\mathbb{R}^n)$ *and* $\tilde{u}_0 \in \mathcal{S}'_{p^+}(\mathbb{R}^n)$ *fulfill* (5.4) *or, equivalently,* (5.5) *and* (5.6)*. Then* $u \in \mathcal{S}'(\mathbb{R}^{1+n})$ *is a solution of the (non-characteristic) Cauchy problem*

$$(\Box + m^2)u = 0, \qquad u|_{x^0=0} = u_0, \qquad \partial_0 u|_{x^0=0} = u_1$$

if and only if $\tilde{u} = u \circ \kappa^{-1} \in \mathcal{S}'(\mathbb{R}^{1+n})$ *is a solution of the (characteristic) Cauchy problem*

$$(\widetilde{\Box} + m^2)\tilde{u} = 0, \qquad \tilde{u}|^*_{x^+=0} = \tilde{u}_0.$$

In this case, u *(respectively* $\tilde{u}$*) is uniquely determined by the initial data* u_0*,* u_1 *(respectively* $\tilde{u}_0$*).*

Example 5.6. Consider the Pauli-Jordan function

$$D_m(x) = \frac{1}{i(2\pi)^3} \int d^4p \, \epsilon(p^0)\delta(p^2 - m^2) e^{i\langle p,x\rangle_{\mathbb{M}}} \in \mathcal{S}'(\mathbb{R}^4).$$

which is a fundamental solution of the Cauchy problem of the Klein-Gordon equation, i.e.,

$$(\Box + m^2)D_m = 0 \quad \text{and} \quad D_m|_{x^0=0} = 0, \quad \partial_0 D_m|_{x^0=0} = \delta(\mathbf{x})$$

Transforming D_m to light cone coordinates we obtain $\widetilde{D}_m = D_m \circ \kappa^{-1}$ which is a solution of the LCKG-equation, i.e., $(\widetilde{\Box} + m^2)\widetilde{D}_m = 0$. Now we can use the transformation law (5.4) to compute the tame restriction $\widetilde{D}_m|^*_{x^+=0}$ directly from the restrictions $D_m|_{x^0=0}$ and $\partial_0 D_m|_{x^0=0}$. Since the Fourier transform of $\delta(\mathbf{x})$ is the constant function $1 \in \mathcal{S}'(\mathbb{R}^3)$ we get

$$\left(\widetilde{D}_m|^*_{x^+=0}\right)^{\sqcap} = \frac{i}{2|p^+|} \left(\nu^*_{>0}1 - \nu^*_{<0}1\right).$$

and hence for all $g \in \mathcal{S}_{p^+}(\mathbb{R}^3)$

$$\left(\left(\widetilde{D}_m|^*_{x^+=0}\right)^{\sqcap}, g\right) = i \int_{p^+>0} \frac{d^3\tilde{\mathbf{p}}}{2p^+} \left(g(\tilde{\mathbf{p}}) - g(-\tilde{\mathbf{p}})\right).$$

Applying the Fourier inversion formula, we obtain

$$\left(\widetilde{D}_m|^*_{x^+=0}, f\right) = \frac{i}{(2\pi)^3} \int_{p^+>0} \frac{d^3\tilde{\mathbf{p}}}{2p^+} \left(f^{\sqcap}(-\tilde{\mathbf{p}}) - f^{\sqcap}(\tilde{\mathbf{p}})\right). \tag{5.7}$$

for all $f \in \mathcal{S}_{\partial_-}(\mathbb{R}^3)$. In [16] we have determined the right-hand side of (5.7) and thus we get finally

$$\widetilde{D}_m|^*_{x^+=0} = \frac{1}{4}(\delta(x_\perp) \otimes \epsilon(x^-)). \tag{5.8}$$

The result (5.8) is related to what is called "quantization on the light cone" [4, 3]. Recall that a real scalar free field $\phi = \phi_m$ obeys the commutator relation $[\phi(x), \phi(y)] = -iD_m(x-y)$ from which one obtains the canonical equal time commutator relations

$$[\phi(0,\mathbf{x}), \phi(0,\mathbf{y})] = 0, \qquad [\phi(0,\mathbf{x}), \pi(0,\mathbf{y})] = i\delta(\mathbf{x} - \mathbf{y})$$

by restricting D_m and $\partial_0 D_m$ to $\{x^0 = 0\}$; as usual we have set $\pi(x) = \partial_0\phi(x)$. In light cone quantum field theory the field ϕ is quantized by demanding commutator relations for fixed LC-time $x^+ = y^+ = 0$, where $x^+ = \frac{1}{\sqrt{2}}(x^0 + x^3)$ and $y^+ = \frac{1}{\sqrt{2}}(y^0 + y^3)$, respectively.

Transforming the covariant field $\phi(x)$ to light cone variables $\widetilde{\phi}(\tilde{x}) = (\phi \circ \kappa^{-1})(\tilde{x})$, the commutator relation of light cone quantum field theory reads [4]

$$[\widetilde{\phi}(0, \tilde{\mathbf{x}}), \widetilde{\phi}(0, \tilde{\mathbf{y}})] = \frac{1}{4i} \left(\delta(x_\perp - y_\perp) \otimes \epsilon(x^- - y^-) \right).$$

Furthermore, in light cone physics it is generally assumed that the canonical field $\widetilde{\pi} = \partial_+ \widetilde{\phi}$ is superfluous and need not be considered through quantization. This is in accordance with our transformation laws (5.4) and (5.5), (5.6) where the tame restriction $\tilde{u}|^*_{x^+=0}$ of a solution $\tilde{u} \in \mathcal{S}'(\mathbb{R}^{1+n})$ of the LCKG-equation determines both $u|_{x^0=0}$ and $\partial_0 u|_{x^0=0}$, and vice verse, where $u = \tilde{u} \circ \kappa$. Now since $[\phi(x), \phi(y)] = -iD_m(x-y)$ we obtain $[\widetilde{\phi}(\tilde{x}), \widetilde{\phi}(y)] = -i\widetilde{D}_m(\tilde{x}-\tilde{y})$. Since $\widetilde{D}_m(\tilde{x})$ has no canonical restriction to $\{x^+ = 0\}$ it is not surprising that one gets into trouble in trying to restrict canonically $\widetilde{\phi}(\tilde{x})$ to $\{x^+ = 0\}$ – this has been an crucial problem in light cone physics [16]. If we consider the "tame" restriction [16] $\widetilde{\phi}(0, \tilde{\mathbf{x}})$ of the covariant LC-field $\widetilde{\phi}(\tilde{x})$ then we obtain the canonical equal LC-time commutator relation as $-i$ times the tame restriction of $\widetilde{D}_m$ at $\tilde{\mathbf{x}} - \tilde{\mathbf{y}}$ which, according to (5.8), equals $\frac{-i}{4}(\delta(x_\perp - y_\perp) \otimes \epsilon(x^- - y^-))$.

6. Conclusions and Outlook

In this paper we have investigated the characteristic Cauchy problem of the Klein-Gordon equation (5.1) which was motivated from light cone quantum field theory. As yet it has been an open question whether one can force uniqueness within the space of all (physical) solutions. We solved this problem by introducing the function spaces $\mathcal{S}_{p^+}(\mathbb{R}^n)$ and $\mathcal{S}_{\partial_-}(\mathbb{R}^n)$ which appear as subspaces of the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ and which are related to each other by the Fourier transformation on $\mathcal{S}(\mathbb{R}^n)$. To see that our consideration also includes the physical case, recall (cf. [15]) that the "physical" solutions which correspond to the positive energy one-particle states of a real scalar free quantum field are of the form

$$\phi(x^0, \mathbf{x}) = \frac{1}{(2\pi)^3} \int \frac{d^3\mathbf{p}}{2\omega(\mathbf{p})} \left(a(\mathbf{p}) e^{-i(x^0\omega(\mathbf{p}) - \mathbf{x}\cdot\mathbf{p})} + a^+(\mathbf{p}) e^{i(x^0\omega(\mathbf{p}) - \mathbf{x}\cdot\mathbf{p})} \right) \tag{6.1}$$

where $a(\mathbf{p}), a^+(\mathbf{p}) \in \mathcal{L}^2(\mathbb{R}^3, d^3\mathbf{p})$; notice that $\mathcal{S}(\mathbb{R}^3)$ is a dense subspace of $\mathcal{L}^2(\mathbb{R}^3, d^3\mathbf{p})$. It follows from (6.1) that the restrictions $\phi|_{x^0=0}$ and $\partial_0\phi|_{x^0=0}$ are in $\mathcal{L}^2(\mathbb{R}^3, d^3\mathbf{p})$ and, by the general result, ϕ is uniquely determined by these restrictions. Going over to light cone coordinates these solutions correspond to solutions (of the LCKG-equation) of the form

$$\begin{aligned} \widetilde{\phi}(x^+, \tilde{\mathbf{x}}) &= (\phi \circ \kappa^{-1})(x^+, \tilde{\mathbf{x}}) = \\ &= \frac{1}{(2\pi)^3} \int_{p^+>0} \frac{d^2\tilde{\mathbf{p}}}{2p^+} \left(b(\tilde{\mathbf{p}}) e^{-i(x^+\tilde{\omega}(\tilde{\mathbf{p}}) + x^-p^+ - x_\perp\cdot p_\perp)} + b^+(\tilde{\mathbf{p}}) e^{i(x^+\tilde{\omega}(\tilde{\mathbf{p}}) + x^-p^+ - x_\perp\cdot p_\perp)} \right) \qquad (6.2) \\ &= \frac{1}{(2\pi)^3} \int \frac{d^3\tilde{\mathbf{p}}}{2|p^+|} \tilde{b}(\tilde{\mathbf{p}}) e^{-i(x^+\tilde{\omega}(\tilde{\mathbf{p}}) + x^-p^+ - x_\perp\cdot p_\perp)} \end{aligned}$$

where $b(\tilde{\mathbf{p}}), b^+(\tilde{\mathbf{p}})$ are in $\mathcal{L}^2(\mathbb{R}^3, \frac{\Theta(p^+)}{2|p^+|} d^3\tilde{\mathbf{p}})$ and $\tilde{b}(\tilde{\mathbf{p}}) = \Theta(p^+) b(\tilde{\mathbf{p}}) + \Theta(-p^+) b^+(-\tilde{\mathbf{p}})$ is in $\mathcal{L}^2(\mathbb{R}^3, d^3\tilde{\mathbf{p}}/2|p^+|)$. This follows immediately from our transformation law in Proposition 4.20 if one takes into account that $\nu^*_{>0}\mathcal{L}^2(\mathbb{R}^3, d^3\mathbf{p}) = \mathcal{L}^2(\mathbb{R}^3, \frac{\Theta(p^+)}{2|p^+|} d^3\tilde{\mathbf{p}})$; notice that $\mathcal{S}_{p^+>0}(\mathbb{R}^3) = \nu^*_{>0}\mathcal{S}(\mathbb{R}^3)$ is a dense subspace of $\mathcal{L}^2(\mathbb{R}^3, \frac{\Theta(p^+)}{2|p^+|} d^3\tilde{\mathbf{p}})$. We obtain from (6.2) that

the tame restriction $\widetilde{\phi}|^*_{x^+=0}$ is the Fourier transform of a function in $\mathcal{L}^2(\mathbb{R}^3, \frac{1}{2|p^+|}d^3\tilde{\mathbf{p}})$. Our general result says that this tame restriction is in $\mathcal{S}'_{\partial_-}(\mathbb{R}^3)$ and determines $\widetilde{\phi}$ uniquely.

It is also possible to generalize the definitions of the spaces $\mathcal{S}_{p^+}(\mathbb{R}^n)$ and $\mathcal{S}_{\partial_-}(\mathbb{R}^n)$ in the following way: Recall that $\mathcal{S}_{p^+}(\mathbb{R}^n)$ is the (complex) vector space of all C^∞-functions such that

$$(6.3) \qquad ||f||_{Q,\alpha} = \sup_{\tilde{\mathbf{p}}\in\mathbb{R}^n\setminus\{p^+=0\}} |Q(\tilde{\mathbf{p}})\partial^\alpha(\tilde{\mathbf{p}})| < \infty \qquad Q \in \mathbb{C}[p^+, p_\perp]_{p^+},\ \alpha \text{ multi-index},$$

topologized by the seminorms in the left-hand side of(6.3); $\mathcal{S}_{p^+}(\mathbb{R}^n)$ is a subspace of $\mathcal{S}(\mathbb{R}^n)$ and the topology on $\mathcal{S}_{p^+}(\mathbb{R}^n)$ coincides with the subspace topology induced by $\mathcal{S}(\mathbb{R}^n)$. At first sight the definition of $\mathcal{S}_{p^+}(\mathbb{R}^n)$ looks artificial. However, the following consideration shows that the definition is as natural as that of $\mathcal{S}(\mathbb{R}^n)$. Let $F(x) \in \mathbb{C}[X]$, $X = (X_1, \ldots, X_n)$, be a complex polynomial. Consider the (affine) variety $V(F) = \{x \in \mathbb{C}^n : F(x) = 0\}$ which is a hypersurface of the affine space $\mathbb{A}^n$. The complement $D(F) = \mathbb{A}^n \setminus V(F)$ is called a distinguished open subset of $\mathbb{A}^n$; $D(F)$ is itself a variety. The ring of regular functions on $D(F)$ is $\mathbb{C}[X]_F$ which consists of formal fractions q/F^r, $q \in \mathbb{C}[X]$, $r \geq 0$. Any element of $q/F^r \in \mathbb{C}[X]_F$ induces canonically a (complex-valued) function $x \mapsto q(x)/F^r(x)$ on $D(F)$. Now let $\mathcal{S}_F(\mathbb{R}^n)$ be the vector space of all C^∞-functions on $D_\mathbb{R}(F) = D(F) \cap \mathbb{R}^n$ such that

$$(6.4) \qquad ||f||^F_{Q,\alpha} = \sup_{x\in D_\mathbb{R}(F)} |Q(x)\partial^\alpha f(x)| < \infty \qquad Q \in \mathbb{C}[X]_F,\ \alpha \text{ multi-index},$$

topologized by the seminorms in the left-hand side of (6.4); we call the dual space $\mathcal{S}'_F(\mathbb{R}^n)$ the space of *F-tempered distributions* [18]. Notice that any element of $\mathcal{S}_F(\mathbb{R}^n)$ can be extended (by zero) to a C^∞-function on $\mathbb{R}^n$. If $F = \text{const.}$ is a constant polynomial then we recover the Schwartz space $\mathcal{S}(\mathbb{R}^n)$, i.e., $\mathcal{S}_F(\mathbb{R}^n) = \mathcal{S}(\mathbb{R}^n)$. Hence $\mathcal{S}_F(\mathbb{R}^n)$ is a natural generalization of the Schwartz space $\mathcal{S}(\mathbb{R}^n)$. In the solution of the characteristic Cauchy problem of the Klein-Gordon equation we have made use of $\mathcal{S}_F(\mathbb{R}^n)$ where $F(\tilde{\mathbf{p}}) = p^+$. If $F_1, F_2 \in \mathbb{C}[X]$ are polynomials such that $D(F_1) \subset D(F_2)$, then canonically $\mathbb{C}[X]_{F_2} \subset \mathbb{C}[X]_{F_1}$ and one obtains $\mathcal{S}_{F_1}(\mathbb{R}^n) \subset \mathcal{S}_{F_2}(\mathbb{R}^n)$. Hence, if especially $F_2 = \text{const.}$ is a constant polynomial then we obtain $\mathcal{S}_F(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$ for any polynomial $F \in \mathbb{C}[X]$. Thus one can consider the preimage of $\mathcal{S}_F(\mathbb{R}^n)$ in $\mathcal{S}(\mathbb{R}^n)$ under the Fourier transformation; this preimage is denoted $\mathcal{S}_{F(\partial)}(\mathbb{R}^n)$. Furthermore, the dual spaces $\mathcal{S}'_F(\mathbb{R}^n)$ and $\mathcal{S}'_{F(\partial)}(\mathbb{R}^n)$ depend functorially on $D(F)$, i.e., they are presheaves. These spaces may be helpful in constructing non-canonical restrictions (and products) of distributions and in studying the characteristic Cauchy problem of partial differential equations in general.

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